Final exam – practice problems

Linear independence

1. Check if the set of vectors $\{\alpha_1, \ldots, \alpha_n\}$ is linearly independent if
   - $\alpha_1 = [1,\ 2,\ 3], \quad \alpha_2 = [3,\ 6,\ 7]$;
   - $\alpha_1 = [4,\ -2,\ 6], \quad \alpha_2 = [6,\ -3,\ 9]$;
   - $\alpha_1 = [2,\ -3,\ 1], \quad \alpha_2 = [3,\ -1,\ 5], \quad \alpha_3 = [1,\ -4,\ 3]$;
   - $\alpha_1 = [2,\ -3,\ 1], \quad \alpha_2 = [3,\ -1,\ 5], \quad \alpha_3 = [1,\ -4,\ 3], \quad \alpha_4 = [1, 1, 3]$;
2. Check if the polynomials $1,\ X,\ X^2, \ldots, X^n$ are linearly independent.
3. Check if the polynomials $\alpha_1 = 2X + 1, \alpha_2 = X^2, \alpha_3 = (X+1)^2$ are linearly independent.
4. Check if the rational functions $\frac{1}{X}, \frac{1}{X-1}, \frac{1}{X-2}, \ldots, \frac{1}{X-n}$ are linearly independent.
5. Check if the functions $f_1(x) = 1,\ f_2(x) = \sin x,\ f_3(x) = \sin 2x$ are linearly independent.
6. Find values of a for which the vectors $\alpha_1 = [1, 2, -1]$, $\alpha_2 = [0, 1, 3]$, $\alpha_3 = [1, 1, 0]$ are linearly independent.
7. Supose that the vectors $\alpha_1, \ldots, \alpha_n$ are linearly independent. Check if the vectors $\beta_1, \ldots, \beta_n$ are also linearly independent, if
   - $\beta_1 = \alpha_1, \quad \beta_2 = \alpha_1 + \alpha_2, \quad \beta_3 = \alpha_1 + \alpha_2 + \alpha_3, \quad \ldots, \quad \beta_n = \alpha_1 + \alpha_2 + \ldots + \alpha_n$;
   - $\beta_1 = \alpha_1 + \alpha_2, \quad \beta_2 = \alpha_2 + \alpha_3, \quad \ldots, \quad \beta_{n-1} = \alpha_{n-1} + \alpha_n, \quad \beta_n = \alpha_n + \alpha_1$.

Systems of linear equations

1. Solve the following systems of equations. $\begin{cases} x + 4y + 10z + 20t = x \\ -6y - 20z - 45t = y \\ 4y + 15z + 36t = z \\ -y - 4z - 10t = t \end{cases}$, $\begin{cases} x + 4y + 10z + 20t = -x \\ -6y - 20z - 45t = -y \\ 4y + 15z + 36t = -z \\ -y - 4z - 10t = -t \end{cases}$,
2. Find bases for subspaces of solutions of the following systems of equations:

$$\begin{cases} X_1 + X_2 - X_3 = 0 \\ 2X_2 - X_4 = 0 \end{cases} ; \qquad \begin{cases} X_1 + X_2 - X_3 = 0 \\ 2X_2 - X_4 = 0 \end{cases} .$$

3. Depending on a find dimensions of subspaces of solutions of the following systems of equations:

$$\begin{cases} 2X_1 + X_2 + X_3 = 0 \\ X_1 + 2X_2 + X_3 = 0 \\ X_1 + X_2 + aX_3 = 0 \end{cases}, \begin{cases} X_1 + 2X_2 + 3X_3 + 4X_4 = 0 \\ 4X_1 + aX_2 + 3X_3 + 4X_4 = 0 \\ 2X_1 + X_2 + 2X_3 + X_4 = 0 \end{cases},$$

4. Solve the following systems of equations: $\begin{cases} 2x - 3y + 5z + 7t = 1 \\ 4x - 6y + 2z + 3t = 2 \\ 2x - 3y - 11z - 15t = 1 \end{cases}$; $\begin{cases} 2x + 5y - 8z = 8 \\ 4x + 3y - 9z = 9 \\ 2x + 3y - 5z = 7 \\ x + 8y - 7z = 12 \end{cases}$; $\begin{cases} 3x + 4y + z + 2t = 3 \\ 6x + 8y + 2z + 5t = 7 \\ 9x + 12y + 3z + 10t = 13 \end{cases}$;

$$\begin{cases} 3x - 5y + 2z + 4t = 2 \\ 7x - 4y + z + 3t = 5 \\ 5x + 7y - 4z - 6t = 3 \end{cases}; \begin{cases} 3x - 2y + 5z + 4t = 2 \\ 6x - 4y + 4z + 3t = 3 \\ 9x - 6y + 3z + 2t = 4 \end{cases}; \begin{cases} 8x + 6y + 5z + 2t = 21 \\ 3x + 3y + 2z + t = 10 \\ 4x + 2y + 3z + t = 8 \\ 3x + 5y + z + t = 15 \\ 7x + 4y + 5z + 2t = 18 \end{cases};$$

$$\begin{cases} x + y + 3z - 2t + 3w = 1 \\ 2x + 2y + 4z - t + 3w = 2 \\ 3x + 3y + 5z - 2t + 3w = 1 \\ 2x + 2y + 8z - 3t + 9w = 2 \end{cases}; \begin{cases} 2x - y + z + 2t + 3w = 2 \\ 6x - 3y + 2z + 4t + 5w = 3 \\ 6x - 3y + 2z + 8t + 13w = 9 \\ 4x - 2y + z + t + 2w = 1 \end{cases}; \begin{cases} 6x + 4y + 5z + 2t + 3w = 1 \\ 3x + 2y + 4z + t + 2w = 3 \\ 3x + 2y - 2z + t \qquad = -7 \\ 9x + 6y + z + 3t + 2w = 2 \end{cases}.$$

5. Solve the following systems of equations over complex numbers:

$$\begin{cases} (1+i)x + 2iy - z = 3+2i \\ (3+i)x + (1-i)y + 4z = 6+i \\ 5x + y - iz = 2 \end{cases}, \qquad \begin{cases} (1+i)x + 2y - iz = 2-3i \\ 3x + iy + (2-i)z = 6+4i \\ (4+i)x + y + 3z = 6+6i \end{cases}.$$

6. Depending on parameters a, b solve the following systems of equations:

$$\begin{cases} x+y+2z=1 \\ x-y+z=0 \\ 2x+ay+2z=b \end{cases}, \begin{cases} ax+y+z=1 \\ x+ay+z=a \\ x+y+az=a^2 \end{cases}, \begin{cases} ax+y+z=4 \\ x+by+z=3 \\ x+2by+z=4 \end{cases}, \begin{cases} ax+by+z=1 \\ x+aby+z=b \\ x+by+az=1 \end{cases}.$$

7. Find systems of equations to which the following are sets of solutions: $Span([2,\ 1,\ 3,\ 1],\ [1,\ 3,\ 4,\ 3],\ [1,\ 0,\ 1,\ 2],\ [0,\ 2,\ 2,\ 4])$, $Span([2,\ 1,\ 3,\ 1],\ [0,\ 1,\ 1,\ 2],\ [1,\ 0,\ 1,\ 2],\ [0,\ 2,\ 2,\ 4])$.
$[1,2,4,4] + Span([1,-1,-3,-1])$
$[1,0,3] + Span([1,2,3],[-2,4,1])$
$[0,1,2] + Span([1,1,1])$.

Matrix multiplication

1. Find the following products of matrices: $\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 0 \\ -1 & 5 \end{bmatrix}$, $\begin{bmatrix} 6 & 4 \\ -2 & 1 \\ 7 & 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$, $\begin{bmatrix} -3 & 4 & 1 \\ 0 & 2 & 8 \\ 1 & 3 & -1 \end{bmatrix}^2$, $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}^3$,
$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T \cdot \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T$, $\begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 3 & 2 \end{bmatrix}^T \cdot \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 3 & 2 \end{bmatrix}$.

2. For $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ i $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ find:
$A^2 + 2AB + B^2$ and $(A+B)^2$; $A^2 - 2AB + B^2$ and $(A-B)^2$; $A^2 - B^2$, $(A-B)(A+B)$ and $(A+B)(A-B)$.

3. Find all 2×2 matrices A such that:
$A\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} A$, $A\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Ranks of matrices

1. Find ranks of the following matrices:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 \\ 1 & 4 & 1 & 1 \\ 5 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 1 & 2 & -1 \\ 0 & 2 & -1 & 1 & 2 \\ 4 & 3 & 2 & -1 & 1 \\ 12 & 9 & 8 & -7 & 3 \\ -12 & -5 & -8 & 5 & 1 \end{bmatrix}, \begin{bmatrix} 8 & 1 & -2 \\ 2 & 7 & 4 \\ 2 & 4 & 2 \\ -1 & -2 & -1 \\ 1 & 5 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 7 & 7 & 9 \\ 7 & 5 & 1 & -1 \\ 4 & 2 & -1 & -3 \\ -1 & 1 & 3 & 5 \end{bmatrix},$$

$$\begin{bmatrix} 4 & 0 & 3 & 2 \\ 1 & -7 & 4 & 5 \\ 7 & 1 & 5 & 3 \\ -5 & -3 & -3 & -1 \\ 1 & -5 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 8 & -4 & 5 & 5 & 9 \\ 1 & -3 & -5 & 0 & -7 \\ 7 & -5 & 1 & 4 & 1 \\ 3 & -1 & 3 & 2 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 4 & -1 & 0 & -1 & 8 \\ 2 & 3 & 7 & 5 & 2 & 3 \\ 3 & 2 & 5 & 7 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 & 2 \\ 1 & 7 & 6 & 6 & 5 & 7 \\ 2 & 1 & 1 & 2 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 5 & -4 & 4 & 0 & 0 & 0 \\ 9 & -7 & 6 & 0 & 0 & 0 \\ 3 & -2 & 1 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ -2 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

2. Find ranks of the following matrices:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 0 & \cdots & 0 & 0 \\ 1 & 3 & 2 & \cdots & 0 & 0 \\ 0 & 1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & 2 \\ 0 & 0 & 0 & \cdots & 1 & 3 \end{bmatrix},$$

$$\begin{bmatrix} a & 1 & 1 & 1 & \cdots & 1 \\ 1 & a & 1 & 1 & \cdots & 1 \\ 1 & 1 & a & 1 & \cdots & 1 \\ 1 & 1 & 1 & a & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & a \end{bmatrix}, \quad \begin{bmatrix} a & 1 & 1 & \cdots & 1 & 1 \\ -1 & a & 1 & \cdots & 1 & 1 \\ -1 & -1 & a & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & a & 1 \\ -1 & -1 & -1 & \cdots & -1 & a \end{bmatrix}, \quad \begin{bmatrix} 1 & n & n & \cdots & n & n \\ n & 2 & n & \cdots & n & n \\ n & n & 3 & \cdots & n & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n & n & n & \cdots & n-1 & n \\ n & n & n & \cdots & n & n \end{bmatrix}.$$

3. Depending on a, find ranks of the following matrices: $\begin{bmatrix} -a & 1 & 2 & 3 & 1 \\ 1 & -a & 3 & 2 & 1 \\ 2 & 3 & -a & 1 & 1 \\ 3 & 2 & 1 & -a & 1 \end{bmatrix}, \begin{bmatrix} a & 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 & 1 \\ 1 & 1 & a & 1 & 1 \\ 1 & 1 & 1 & a & 1 \\ 1 & 1 & 1 & 1 & a \end{bmatrix}.$

4. Find ranks of the following matrices over complex numbers: $\begin{bmatrix} 1+i & 1+i & 1-i \\ 1-i & -1+i & 1+3i \\ 1 & i & 1+i \end{bmatrix}, \quad \begin{bmatrix} 1-i & i & -1 \\ 1 & 0 & 2i \\ i & 2-i & 1+i \end{bmatrix}.$