Name: **Perm number:**

Midterm – take-home part

Due date: Wednesday, May 12th

Let $(G, *, e)$ be a group. A subset H of the set G that contains the element e and such that $(H, *, e)$ is a group itself (written $H < G$) is called a **subgroup.**

(1) (**5 points**) Check that $\mathbb{Z} < \mathbb{R}$, $\mathbb{R}^* < \mathbb{C}^*$, $SL(n, K) < GL(n, K)$.

(2) (**20 points**) Let $(G, *, e)$ be a group, and let $\emptyset \neq H \subset G$. Prove that the following three conditions are equivalent:

- $H < G$,
- H has the following properties:
  - $e \in H$,
  - $\forall a, b \in H (a * b \in H)$,
  - $\forall a \in H (a^{-1} \in H)$.
- H has the following property:
  - $\forall a, b \in H (a * b^{-1} \in H)$.

(3) (**5 points**) Check that $\mathbb{C}(n) < \mathbb{C}^*$, $\{0, 2, 4\} < \mathbb{Z}$, $2\mathbb{Z} = \{2k : k \in \mathbb{Z}\} < \mathbb{Z}$.

(4) (**20 points**) Let $(G, *, e)$ be a group, let $H_1 < G$ and $H_2 < G$. Show that $H_1 \cap H_2 < G$. Is it true that $H_1 \cup H_2 < G$? Either prove the statement, or give a counterexample.

Let $(G, *, e)$ be a group, let $A \subset G$. The intersection of all subgroups of G that contain the set A is called the **subgroup generated by** A and denoted by $< A >$.

Every subset A with the property that $< A >= G$ is called the **set of generators** of G. If $A = \{a_1, a_2, \ldots, a_n\}$, we write $< a_1, a_2, \ldots, a_n >$ to denote $< A >$.

If there exist elements $a_1, a_2, \ldots, a_n \in G$ such that $G =< a_1, a_2, \ldots, a_n >$, we say that G is **finitely generated**.

(5) (**25 points**) Let $(G, *, e)$ be a group, let $A \subset G$. Prove that

$$< A >= \{a_1^{k_1} * a_2^{k_2} * \ldots * a_n^{k_n} : n \in \mathbb{N}, k_i \in \mathbb{Z}, a_i \in A\}$$

(Hint: Let $M = \{a_1^{k_1} * a_2^{k_2} * \ldots * a_n^{k_n} : n \in \mathbb{N}, k_i \in \mathbb{Z}, a_i \in A\}$. You need to show that $< A >= M$. To show that $< A >\subset M$, prove that $M < G$ and that $A \subset M$ (why is this enough?). To show that $< A >\supset M$ use induction with respect to n)

(6) (**25 points**) Show that every finitely generated subgroup of $\mathbb{Q}$ can be generated by only one element. Find an element $a \in \mathbb{Q}$ such that $< a >=< \frac{2}{3}, \frac{4}{5} >$.