Project 10 - Cardinality.

We say that a function is a **bijection** if it is both one-to-one and onto.

Check that the following functions are bijections:

(1) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $f(n, m) = 2^n \cdot (2m + 1) - 1$
(2) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $\binom{f(n,m)=}{n+m+12+n}$
(3) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $f(n, m) = \sum_{k=1}^{n+m-1} k + m - 1$
(4) $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$, $f(n, m) = \frac{n}{m+1}$

We say that two sets X and Y have **the same cardinal number**, written $|X| = |Y|$, if there is a bijection between X and Y.

Check that the following sets have the same cardinal number as the set $\mathbb{N}$:

(1) $\mathbb{N} \times \mathbb{N}$
(2) $\mathbb{Z}$
(3) $\mathbb{Q}$
(4) every infinite subset of $\mathbb{N}$

Sets that have the same cardinal number as the set $\mathbb{N}$ are said to be **countable**. Infinite sets that are not countable are called **uncountable**. Sets that are countable are also said to have the cardinal number equal to $\aleph_0$ (aleph zero).

(1) Show that the set $\mathbb{R}$ is uncountable.
(2) Show that $|\mathbb{R}| = |\mathbb{R} \times \mathbb{R}|$

Sets that have the same cardinal number as the set $\mathbb{R}$ are said to have the cardinal number equal to $\mathfrak{c}$ (continuum).

If there is a one-to-one function $f : X \to Y$, then we say that X has the cardinal number no greater than Y and write $|X| \leq |Y|$.

If $|X| \leq |Y|$ but $|X| \neq |Y|$, then we say that X has the cardinal number less than Y and write $|X| < |Y|$.

Prove that:

(1) if $|A| = |B|$, $|C| = |D|$, and $A \cap B = C \cap D = \emptyset$, then $|A \cup B| = |C \cup D|$
(2) if $|A| = |B|$, then $|P(A)| = |P(B)|$ ($P(X)$ denotes the set of all subsets of X)
(3) $|P(A)| = |\{0, 1\}^A|$ (Y^X denotes the set of all functions $X \to Y$)
(4) if $|A| = |C|$ and $|B| = |D|$, then $|A^B| = |C^D|$
(5) if $B \cap C = \emptyset$, then, for every A, $|A^{B \cup C}| = |A^B \times A^C|$
(6) $|(A^B)^C| = |A^{B \times C}|$
(7) (Cantor Theorem) $|A| < |P(A)|$
(8) (Cantor-Bernstein Theorem) if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.
(9) $|P(\mathbb{N})| = |\mathbb{R}|$

In view of Problems 7 and 9, it makes sense to ask whether there are sets S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$. This question is now known as the **continuum hypothesis**.