Project 8 - Functions.

Just as we defined relations on a set X to be a subset of the Cartesian product $X \times X$, we can define relations between elements of two different sets X and Y as subsets of the Cartesian product $X \times Y$.

Let X and Y be two nonempty sets. A **function** F is the ordered triple (F, X, Y), where $F \subset X \times Y$ is a relation between the elements of X and Y which is **single-valued**, that is

$$\forall (x_1, y_1), (x_2, y_2) \in F[(x_1 = x_2) \Rightarrow (y_1 = y_2)]$$

We usually write $F : X \to Y$, and call X the **domain** of F, and Y the **codomain** of F.

A function is **one-to-one** if

$$\forall (x_1, y_1), (x_2, y_2) \in F[(y_1 = y_2) \Rightarrow (x_1 = x_2)]$$

A function is **onto** if

$$\forall y \in Y \exists x \in X (x, y) \in F.$$

Check if the following relations are functions. If so, are they one-to-one? Onto?

(1) $F \subset \mathbb{N} \times \mathbb{N}$, $aFb \Leftrightarrow ab = 12$
(2) $F \subset \mathbb{N} \times \mathbb{N}$, $aFb \Leftrightarrow a + b = 15$
(3) $F \subset \mathbb{N} \times \mathbb{Z}$, $aFb \Leftrightarrow 2a - b = 1$
(4) $F \subset \mathbb{R} \times \mathbb{R}$, $aFb \Leftrightarrow a = |b|$
(5) $F \subset \mathbb{R} \times \mathbb{R}$, $aFb \Leftrightarrow a = b^2$
(6) $F \subset (0, +\infty) \times \mathbb{R}$, $aFb \Leftrightarrow a = b^2$
(7) $F \subset (\mathbb{N} \setminus \{1\}) \times (\mathbb{N} \setminus \{1\}$, $aFb \Leftrightarrow b|a \wedge [\forall m \in \mathbb{N} \setminus \{1\}(m|a \to m \geq b)]$

Instead of writing $(x, y) \in F$, we often write $F(x) = y$. Usually it is convenient to define functions by describing the corresponding value y to a given x.

If a function $F : X \to Y$ is one-to-one, we can define the **inverse function** F^{-1} as follows:

$$(y, x) \in F^{-1} \Leftrightarrow (x, y) \in F$$

Equivalently:

$$F^{-1}(y) = x \Leftrightarrow F(x) = y.$$

For a given function f check if it is one-to-one, onto, find its inverse (if it exists), and find $f(A)$ and $f^{-1}(B)$ if:

(1) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -2x$, $A = B = \{1\}$
(2) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 1$, $A = \{2k : k \in \mathbb{Z}\}$, $B = \{0\}$
(3) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$, $A = [0, \frac{\pi}{4}]$, $B = [1, 2]$
(4) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \frac{x+1}{x-1} \text{ for } x \neq 1 \\ 1 \text{ for } x = 1 \end{cases}$, $A = [0, 1]$, $B = \mathbb{N}$
(5) $f : [\frac{1}{2}\pi, \frac{3}{2}\pi] \to [-1, 1]$, $f(x) = \sin x$, $A = \{\pi\}$, $B = [-1, 0]$

Let X, Y, Z be sets, and let $f : X \to Y$, $g : Y \to Z$ be functions. The **composition** of f and g is the function $g \circ f : X \to Z$ which is defined by the rule

$$(g \circ f)(x) = g(f(x))$$

The domain of $g \circ f$ is the set of all x in the domain of f such that $f(x)$ is in the domain of g.

Find the functions $f \circ g$, $g \circ f$, and their domains.

(1) $f(x) = 2x^2 - x$, $g(x) = 3x + 2$
(2) $f(x) = 1 - x^3$, $g(x) = 1/x$
(3) $f(x) = \sin x$, $g(x) = 1 - \sqrt{x}$
(4) $f(x) = 1 - 3x$, $g(x) = 5x^2 + 3x + 2$
(5) $f(x) = x + \frac{1}{x}$, $g(x) = \frac{x+1}{x+2}$
(6) $f(x) = \sqrt{2x + 3}$, $g(x) = x^2 + 1$