Project 7 - Relations.

The **Cartesian product** of two sets A and B is defined as:

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

(1) Describe the set $\{0, 1, 2\} \times \{2, 3, 4\}$.
(2) Describe the set $\{0, 1\} \times \{a, b, c, d\}$.
(3) Describe the set $\{x_1, x_2, \ldots, x_n\} \times \{y_1, y_2, \ldots, y_m\}$.

A **relation** R on a set A is a subset of the Cartesian product $A \times A$. If $(a, b) \in R$, we often write aRb.

We say that a relation R is **reflexive** if:

$$\forall a \in A(aRa)$$

We say that a relation R is **symmetric** if:

$$\forall a, b \in A(aRb \to bRa)$$

We say that a relation R is **antisymmetric** if:

$$\forall a, b \in A(aRb \wedge bRa \to a = b)$$

We say that a relation R is **asymmetric** if:

$$\forall a, b \in A(aRb \to \neg bRa)$$

We say that a relation R is **transitive** if:

$$\forall a, b, c \in A(aRb \wedge bRc \to aRc)$$

We say that a relation R is **linear** if:

$$\forall a, b \in A(aRb \vee bRa)$$

We say that a relation R is **trichotomous** if:

$$\forall a, b \in A(!aRb \vee !bRa \vee !a = b)$$

(1) Let $A = \{a, b, c, d\}$, let $R = \{(a, a), (a, b), (b, b)\}$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(2) Let $A = \{a, b, c, d\}$, let $R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a)\}$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(3) Let $A = \{a, b, c, d\}$, let $R = \{(a, b), (a, c), (b, c), (c, c), (a, a), (b, b)\}$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(4) Let $A = \{0, 1, 2\}$, let $aRb \Leftrightarrow a < b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(5) Let $A = \{1, \ldots, 10\}$, let $aRb \Leftrightarrow a|b \wedge a \neq b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(6) Let $A = \{1, 2, 3, 4\}$, let $aRb \Leftrightarrow 2|a + b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(7) Let $A = \mathbb{Z}$, let $aRb \Leftrightarrow 3|a - b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(8) Let $A = \mathbb{N}$, let $aRb \Leftrightarrow 2|a + b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(9) Let $A = \mathbb{N}$, let $aRb \Leftrightarrow a \neq 0 \wedge a|b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(10) Let $A = \mathbb{N} \setminus \{0\}$, let $aRb \Leftrightarrow a|b \wedge a \neq b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(11) Let $A = \mathbb{R}$, let $aRb \Leftrightarrow a^2 = b^2$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(12) Let $A = \mathbb{R}$, let $aRb \Leftrightarrow a^2 \neq b^2$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(13) Let $A = \mathbb{C}$, let $aRb \Leftrightarrow |a| < |b|$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(14) Let $A = \mathbb{Z}$, let $aRb \Leftrightarrow |a| + |b| \neq 4$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(15) Let $A = \mathbb{R}$, let $aRb \Leftrightarrow a - b \in \mathbb{Q}$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?
(16) Let $A = \mathbb{Q} \times \mathbb{Q}$, let $(a, b)R(c, d) \Leftrightarrow ad = bc$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous?

An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.

For a set A, an equivalence relation R, and an element $a \in A$ we define the **equivalence class** of a with respect to R as the set:

$$[a]_R = \{b \in A : aRb\}$$

For a given set A and a relation R check that R is an equivalence relation on A and describe its equivalence classes:

(1) $A =$ the set of even integers, $aRb \Leftrightarrow 3|a - b$
(2) $A = \mathbb{N}$, $aRb \Leftrightarrow 2|a + b$
(3) $A = \mathbb{Z}$, $aRb \Leftrightarrow 5|a - b$
(4) $A = \mathbb{Z}$, $aRb \Leftrightarrow p|a - b$, where p is a fixed prime number
(5) $A = \{1, 2, \ldots, 16\}$, $aRb \Leftrightarrow 4|a^2 - b^2$

(6) $A =$ polynomials in one variable t with coefficients from $\mathbb{Q}$, $a(t)Rb(t) \Leftrightarrow \exists p, q \in \mathbb{Q} a(t) - b(t) = pt + q$
(7) $A = M(2, \mathbb{R})$, $ARB \Leftrightarrow \det A = \det B$
(8) $A =$ polynomials in one variable t with coefficients from $\mathbb{R}$, $a(t)Rb(t) \Leftrightarrow a(t)b(t)$ is of even degree

A **partition** of a set A is a collection of subsets $A_1, A_2, \ldots, A_n$ such that each element of A lies in exactly one of these subsets.

(1) Let A be a set, and let R be an equivalence relation on A. Show that the equivalence classes of R form a partition of A.
(2) Let A be a set, and let R_1, R_2 be two equivalence relations on A. Check if $R_1 \cap R_2$ is an equivalence relation on A.
(3) Let A be a set, and let R_1, R_2 be two equivalence relations on A. Check if $R_1 \cup R_2$ is an equivalence relation on A.
(4) Let A be a set, and let R be an equivalence relation on A. Check if $A \times A \setminus R$ is an equivalence relation on A.