Project 7 - Field of complex numbers.

A **complex number** can be represented by an expression of the form $a+bi$, where a,b are real numbers and i is a symbol with the property that $i^2=-1$. The **real part** of the number $a+bi$ is a, whilst the **imaginary part** is b.

The sum and the product of two complex numbers are defined as follows:

$$(a+bi)+(c+di)=(a+c)+(b+d)i,$$

$$(a+bi)\cdot(c+di)=(ac-bd)+(ad+bc)i.$$

Find real numbers a and b such that

(1) $a(2+3i)+b(4-5i)=6-2i$
(2) $a(-\sqrt{2}+i)+b(3\sqrt{2}+5i)=8i$
(3) $a(4-3i)^2+b(1+i)^2=7-12i$
(4) $\frac{a}{2-3i}+\frac{b}{3+2i}=1$
(5) $a\frac{2+i}{3-i}+b(\frac{4-i}{1-3i})^2=1+i$
(6) $\frac{2a-3i}{5-3i}+\frac{3a+2i}{3-5i}=0$

Solve the following equations. Write your answer in the form $z=x+iy$, where x and y are real numbers.

(1) $(a-bi)z=a+bi$
(2) $(1-i\tan\alpha)z=1+i\tan\alpha$
(3) $(a+bi)^2(1-z)+(a-bi)^2(1+z)=0$
(4) $(a+bi)z=(2a+3b)+(2b-3a)i$
(5) $(1-i)z=(2a-b)-(2a+b)i$

Solve the following systems of equations. Write your answers in the form $z=x+iy$, $w=u+iv$ where x,y,u,v are real numbers.

(1) $\begin{cases} 2(2+i)z-i(3+2i)w=5+4i \\ (3-i)z+2(2+i)w=2(1+3i) \end{cases}$
(2) $\begin{cases} (4-3i)z+2(2+i)w=2(1+3i) \\ (2-i)z-(2+3i)w=-(1+i) \end{cases}$
(3) $\begin{cases} (2+i)z+(2-i)w=6b-a+(2a-3b)i \\ (1-i)z+(3+i)w=a+9b+(a+3b)i \end{cases}$
(4) $\begin{cases} \frac{z}{2-i}+\frac{w}{1+i}=2 \\ \frac{5z}{(2-i)^2}+\frac{2w}{(1+i)^2}=3 \end{cases}$

Any complex number $z=a+bi$ can be considered as a point (a,b) on a real plane, and any such point can be represented by polar coordinates:

$$a=r\cos\theta, b=r\sin\theta.$$

Thus, we can write any complex number z in the form

$$z=r(\cos\theta+i\sin\theta).$$

The **modulus**, or **absolute value**, $|z|$ of a complex number $z=a+bi$ is its distance from the origin, $|z|=\sqrt{a^2+b^2}$. The angle θ is called the **argument** of z.

Write the following numbers in polar form:

(1) $1,\ -1,\ i,\ -i,$
(2) $1+i,\ 1-i,\ -1+i,\ -1-i,$
(3) $1+i\sqrt{3},\ 1-i\sqrt{3},\ -1+i\sqrt{3},\ -1-i\sqrt{3}$
(4) $\sqrt{3}+i,\ \sqrt{3}-i,\ -\sqrt{3}+i,\ -\sqrt{3}-i,$
(5) $\sqrt{6}+\sqrt{2}+i(\sqrt{6}-\sqrt{2}),\ \sqrt{6}-\sqrt{2}+i(\sqrt{6}+\sqrt{2})$
(6) $\sqrt{5}+1+i\sqrt{10-2\sqrt{5}},\ \sqrt{10-2\sqrt{5}}+i(\sqrt{5}+1)$
(7) $\sqrt{5}-1+i\sqrt{10+2\sqrt{5}},\ \sqrt{10+2\sqrt{5}}+i(\sqrt{5}-1)$
(8) $\sqrt{2+\sqrt{2}}+i\sqrt{2-\sqrt{2}},\ \sqrt{2-\sqrt{2}}+i\sqrt{2+\sqrt{2}}$
(9) $\sqrt{2+\sqrt{3}}+i\sqrt{2-\sqrt{3}},\ \sqrt{2-\sqrt{3}}+i\sqrt{2+\sqrt{3}}$

Let

$$z_1=r_1(\cos\theta_1+i\sin\theta_1), z_2=r_2(\cos\theta_2+i\sin\theta_2)$$

be two complex numbers written in polar form. Then

$$z_1\cdot z_2=r_1r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)),$$

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)).$$

Perform the following multiplications and divisions using polar form:

(1) $(1+i)(1+i\sqrt{3}),$
(2) $(\sqrt{3}+i)(\sqrt{6}+\sqrt{2}+i(\sqrt{6}-\sqrt{2}))$

(3) $[\sqrt{6}+\sqrt{2}+i(\sqrt{6}-\sqrt{2})](\sqrt{5}+1+i\sqrt{10-2\sqrt{5}})(1+i)$
(4) $\frac{1+i}{1+i\sqrt{3}}$
(5) $\frac{\sqrt{6}+\sqrt{2}+i(\sqrt{6}-\sqrt{2})}{\sqrt{3}+i}$
(6) $(1+i)^{10}$
(7) $(1+i\sqrt{3})^{15}$
(8) $[\sqrt{6}+\sqrt{2}+i(\sqrt{6}-\sqrt{2})]^6$
(9) $\frac{(\sqrt{5}+1+i\sqrt{10-2\sqrt{5}})^8}{(1+i)^5(1-i\sqrt{3})^4}$

De Moivre's Theorems. If $z = r(\cos\theta + i\sin\theta)$, and n is a positive integer, then

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Moreover, z has the n distinct n-th roots

$$w_k = r^{1/n}(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}),$$

where $k \in \{0, 1, 2, \ldots, n-1\}$.

Find the polar form of the following complex numbers:

(1) i^n
(2) $(1+i)^n$
(3) $(1+i\sqrt{3})^n$
(4) $(\sqrt{6}+\sqrt{2}+i(\sqrt{6}-\sqrt{2}))^n$
(5) $(\frac{1+i}{1+i\sqrt{3}})^n$
(6) $(\frac{2-i\sqrt{6}}{1-i})^n$
(7) $[\frac{(1+i)(1+i\sqrt{3})}{\sqrt{5}+1+i\sqrt{10-2\sqrt{5}}}]^n$

Find the square roots of the following numbers. Write your answer in the form $z = x + iy$, where x and y are real numbers.

(1) i, $-i$
(2) $8+6i$, $8-6i$, $-8+6i$, $-8-6i$
(3) $3+4i$, $3-4i$, $-3+4i$, $-3-4i$
(4) $11+60i$, $11-60i$, $-11+60i$, $-11-60i$
(5) $15+8i$, $15-8i$, $-15+8i$, $-15-8i$
(6) $1+i\sqrt{3}$, $1-i\sqrt{3}$, $-1+\sqrt{3}$, $-1-i\sqrt{3}$
(7) $2+3i$, $2-3i$, $-2+3i$, $-2-3i$

Write as $x+iy$ the following numbers:

(1) $\sqrt[4]{16}$
(2) $\sqrt[4]{-1}$
(3) $\sqrt[4]{i}$
(4) $\sqrt[4]{2-i\sqrt{12}}$

Solve the following quadratic equations:

(1) $z^2 - 3z + 3 + i = 0$
(2) $z^2 + (1+4i)z - (5+i) = 0$
(3) $(4-3i)z^2 - (2+11i)z - (5+i) = 0$
(4) $z^2 + 2(1+i)z + 2i$