Project 5 - Real numbers.

Integers.

We will denote by $\mathbb{N}$ the set of all natural numbers, and by $\mathbb{Z}$ the set of all integers.
Let $a, b \in \mathbb{Z}$. We say that a divides b, denoted $a|b$, if, for some integer c, $a \cdot c = b$.

(1) (division with remainder) Let a be a positive integer. Then for any $b \in \mathbb{Z}$ there exist q, r such that
$$b = qa + r \text{ and } 0 \le r < a$$
(2) Let $a, b, d \in \mathbb{Z}$ and suppose that $d|a$ and $d|b$. Then, for any $m, n \in \mathbb{Z}$, $d|(ma + nb)$.
(3) (Euclidean algorithm) Let $a, b \in \mathbb{Z}$. We construct the numbers $q_1, q_2, \ldots, q_{n+1}$ and $r_1, r_2, \ldots, r_n$ by performing the following divisions with remainders:
$$\begin{gathered} b = q_1 a + r_1 \text{ where } 0 \le r_1 < a, \\ a = q_2 r_1 + r_2 \text{ where } 0 \le r_2 < r_1, \\ r_1 = q_3 r_2 + r_3 \text{ where } 0 \le r_3 < r_2, \\ \vdots \\ r_{n-3} = q_{n-1} r_{n-2} + r_{n-1} \text{ where } 0 \le r_{n-1} < r_{n-2}, \\ r_{n-2} = q_n r_{n-1} + r_n \text{ where } 0 \le r_n < r_{n-1}, \\ r_{n-1} = q_{n+1} r_n + 0. \end{gathered}$$
Then r_n is the greatest common divisor of a, b
(4) Let $a, b \in \mathbb{Z}$, let $d = gcd(a, b)$. Then there are integers s, t such that
$$d = sa + tb.$$
(5) If $a = 17$ and $b = 29$, find $d = gcd(a, b)$ and the integers s, t such that $d = sa + tb$.
(6) If $a = 713$ and $b = 552$, find $d = gcd(a, b)$ and the integers s, t such that $d = sa + tb$.
(7) If $a = 299$ and $b = 345$, find $d = gcd(a, b)$ and the integers s, t such that $d = sa + tb$.
(8) Let $a, b, c \in \mathbb{Z}$. If a and c are coprime, and if $c|ab$, then $c|b$.
(9) Let $a, b \in \mathbb{Z}$. Let p be a prime number, and let $p|ab$. Then either $p|a$ or $p|b$.
(10) Let $a_1, \ldots, a_n \in \mathbb{Z}$. Let p be a prime number, and let $p|a_1 \cdot \ldots \cdot a_n$. Then p divides exactly one of the integers $a_1, \ldots, a_n$.

Fundamental theorem of arithmetics: Every natural number ≥ 2 has a unique decomposition as a product of prime numbers.

Rational numbers.

We will denote by $\mathbb{Q}$ the set of all rational numbers. Real numbers that are not rational are called irrational numbers.

(1) Prove that $\sqrt{2}$ is irrational.
(2) Prove that $\sqrt{3}$ is irrational.
(3) Prove that $\sqrt{p}$ is irrational, where p is a prime number.
(4) Prove that $\sqrt[m]{n}$ is rational if and only if n is a m-th power.

Real numbers.

We will denote by $\mathbb{R}$ the set of all real numbers.