Project 3 - Quantifiers.

In propositional logic we deal with with simple **propositional formulas** such as $p =$ "in France people speak French" or $q =$ "7 is a prime number". We will now extend this theory to a type of logic where we will be also dealing with **predicates** and **quantification**, that is we will allow a formula to depend on variables that will determine its truth value. For example, we will consider formulas such as $P(x) =$ "x is a prime number", that will have the truth value 1 if $x = 7$, and the truth value 0 if, for example, $x = 4$.

More precisely, by a **predicate** (or **propositional function**[1]) we understand a type of syntactic formula that depends on some variables, is well-formed and can be assigned a truth value. Each variable in a predicate is assumed to belong to a **domain of discourse**. For example, in the predicate "x is a prime" x represents some integer, so the domain of discourse is the set of all integers. Note that for a predicate $P(x)$ with the domain of discourse X we have:

$$a \in \{x \in X : P(x)\} \Leftrightarrow \omega(P(a)) = 1 \wedge a \in X.$$

If $P(x)$ is a predicate with the domain of discourse X, we write

$$\forall x \in X P(x) \text{ or } \bigwedge_{x \in X} P(x)$$

to indicate that for <u>all</u> elements $x \in X$ the formula $P(x)$ has the truth value 1. The symbol $\forall$ (or $\bigwedge$) is called the **universal quantifier.**

Similarly, if $P(x)$ is a predicate with the domain of discourse X, we write

$$\exists x \in X P(x) \text{ or } \bigvee_{x \in X} P(x)$$

to indicate that for <u>some</u> element $x \in X$ the formula $P(x)$ has the truth value 1. The symbol $\exists$ (or $\bigvee$) is called the **existential quantifier.**

Illustrate the following theorems with examples:

(1) $\neg \forall x \in X P(x) \Leftrightarrow \exists x \in X \neg P(x)$
(2) $\neg \exists x \in X P(x) \Leftrightarrow \forall x \in X \neg P(x)$
(3) $\forall x \in X P(x) \wedge Q(x) \Leftrightarrow (\forall x \in X P(x)) \wedge (\forall x \in X Q(x))$
(4) $\exists x \in X P(x) \vee Q(x) \Leftrightarrow (\exists x \in X P(x)) \vee (\exists x \in X Q(x))$
(5) $\exists x \in X \forall y \in X \forall P(x, y) \Rightarrow \forall y \in X \exists x \in X P(x, y)$
(6) $(\forall x \in X P(x)) \vee (\forall x \in X Q(x)) \Rightarrow \forall x \in X P(x) \vee Q(x)$
(7) $\exists x \in X P(x) \wedge Q(x) \Rightarrow (\exists x \in X P(x)) \wedge (\exists x \in X Q(x))$

Illustrate with the appropriate examples why the following statements are false:

(1) $\forall x \in X \exists y \in X \forall P(x, y) \Rightarrow \exists y \in X \forall x \in X P(x, y)$
(2) $\forall x \in X P(x) \vee Q(x) \Rightarrow (\forall x \in X P(x)) \vee (\forall x \in X Q(x))$
(3) $(\exists x \in X P(x)) \wedge (\exists x \in X Q(x)) \Rightarrow \exists x \in X P(x) \wedge Q(x) \Rightarrow$

A **mathematical theory** is a set of sentences that is closed under logical deduction. We usually distinguish certain subset of "basic" sentences that we call **axioms**. Also, to simplify notation, we usually give names to various objects and thus formulate **definitions**. Sentences that are logical consequences of axioms are then called **theorems**.

First order logic uses only discrete variables; although in the above notation we quantify variables over sets, in the considered examples we always assume domains of discourse to be fixed in one way or another. Theories formlated infrst order logic have certain limitations, for example it's not possible to express finiteness of a set in a first order language. Therefore we introduce the concept of **second order logic**, where variables can range over sets of individuals.

In this course we will have a closer look at the theory of natural numbers based on Peano axioms. This is an example of a second order theory – there exist theories of natural numbers that utilize only first order logic (such as Robinson's arithmetics), although they are usually weaker that Peano's theory (that is not all theorems that are provable in Peano theory are provable)

[1]The term "propositional function", introduced by Johnsonbaugh, is somewhat obsolete, and in most modern books the term "predicate" is used.