Project 2 - Foundations of set theory.

In axiomatic set theory the notion of **a set** is taken as an undefined primitive. The common "definition" of a set as a collection of distinct objects considered as an object itself turns out to be inadequate and leads to certain paradoxes (we will discuss that in some more detail later). However, for what we are doing here, the definition of a set as a collection of things should be good enough.

If a is an element of the set A, we write

$$a \in A.$$

If a is not an element of the set A, we write:

$$a \notin A.$$

By the **set of elements having a property** $P(x)$ we understand the set of all, and only those elements that have the property $P(x)$. We denote this set by

$$\{x : W(x)\}.$$

A set whose elements are $a_1, a_2, \ldots, a_n$, and only these elements, will be called a **finite set** and denoted by

$$\{a_1, a_2, \ldots, a_n\}.$$

A **singleton** set is a set containing only one element:

$$\{a\}.$$

An **empty set** is a set that does not contain any elements, and denoted by

$$\emptyset$$

A set that is neither empty not finite is called **infinite**. We'll talk some more about infinite sets later.

If every element of a set A is also an element of a set B, we can that A **is contained in** B and write

$$A \subset B.$$

We say that two sets A and B are **equal** if they are contained in each other:

$$A = B \leftrightarrow A \subset B \wedge B \subset A.$$

The set of all elements that belong both to a set A and a set B is called the **intersection** of A and B, and denoted

$$A \cap B.$$

The set of all elements that belong either to a set A or a set B is called the **union (sum)** of A and B, and denoted

$$A \cup B.$$

The set of all elements that belong to a set A but don't belong to a set B is called the **difference** between A and B, and denoted

$$A \setminus B.$$

Denote by p the sentence "$x \in A$", and by q the sentence "$x \in B$". Then

$A \subset B$	corresponds to	$p \to q$
$A = B$	corresponds to	$p \leftrightarrow q$
$A \cap B$	corresponds to	$p \wedge q$
$A \cup B$	corresponds to	$p \vee q$
$A \setminus B$	corresponds to	$p \wedge \neg q$

Using the above observation prove or disprove the following statements. Illustrate each theorem with a diagram.

(1) $(A \subset B) \wedge (B \subset C) \to (A \subset C)$
(2) $A \cap (B \cap C) = (A \cap B) \cap C$
(3) $A \cup (B \cup C) = (A \cup B) \cup C$
(4) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(5) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(6) $A \cup (A \cap B) = A$
(7) $A \cap (A \cup B) = A$
(8) if $A \subset B$, then $A \cup (B \setminus A) = B$
(9) $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$
(10) $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$
(11) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$
(12) $(A \cup B \cup C) \setminus (A \cup B) = C$

If a finite set A has n elements, we write $|A| = n$. Note that

$$|A \cup B| = |A| + |B| - |A \cup B|.$$

(1) Prove that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

(2) Find a formula for $|A \cup B \cup C \cup D|$. Then generalize to $|A_1 \cup A_2 \cup \ldots \cup A_n|$.

(3) Out of a total of 30 students, 19 are doing mathematics, 17 are doing music, and 10 are doing both. How many are doing neither?

(4) How many integers are there between 1000 and 999 that contain the digits 0, 8, and 9 at least once each?

(5) 73% of British people like cheese, 76% like apples, and 10% like neither. What percentage like both cheese and apples?

(6) In a class of 30 students, 16 cheer for Manchester United, 17 cheer for Liverpool, and 14 for Chelsea; also 8 cheer for both Man United and Liverpool (wonder how that's possible...), 7 for Man and Chelsea, and 9 for Liverpool and Chelsea. How many cheer for all three teams?

(7) On a plane there are 9 boys, 5 American children, 9 men, 7 non-American boys, 14 Americans, 6 American males, and 7 non-American females. How many people are there on the plane altogether?

(8) Find the number of integers between 1 and 5000 that are divisible by neither 3 nor 4.

(9) Find the number of integers between 1 and 5000 that are divisible by neither 3 nor 4 nor 5.

(10) Find the number of integers between 1 and 5000 that are divisible by one or more of the numbers 4, 5 and 6.