**Project 1** - Natural Deduction.

A **sentence** (also called a **propositional formula, propositional expression, sentential formula** is a type of syntactic formula which is well-formed and has a truth value. We shall usually denote sentences by small letters $p, q, r, \ldots$. A truth value of a formula $p$ will be denoted by $\omega(p)$. $\omega(p) = 1$ means that the sentence $p$ is true, while $\omega(p) = 0$ means that the sentence $p$ is false. For example, the sentence

7 is a prime number

is a true sentence in the arithmetics of integers, and thus has the truth value 1. The sentence

7 is an even number

is false, so its truth value will be 0. In mathematical logic, we don't really bother with determining whether a given sentence has truth value 0 or 1 – instead we will be investigating truth values of sentences combined from other sentences whose truth values are known.

A **negation** of a sentence $p$ is the sentence stating "not true that $p$" and denoted by

$$\neg p$$

For example, if $p$ is the following sentence

7 is an even number

then $\neq p$ is the following one:

not true that 7 is an even number.

The truth value of $p$ here is 0, and the truth value of $\neq p$ is 1. Truth values of negation can be characterized by the following table:

| $p$ | $\neg p$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

Two sentences $p$ and $\neg p$ will be called **contradictory.**

**Law of non-contradiction** states that two contradictory sentences cannot be simultaneously true:

$$\omega(p \text{ and } \neg p) = 0$$

**Law of excluded middle (tertium non datur)** states that out of two contradictory sentences at least one is always true:

$$\omega(p \text{ or } \neg p) = 1$$

**Law of double negation** states that the sentences $p$ and $\neg\neg p$ always have the same truth value:

$$\omega(p) = \omega(\neg\neg p)$$

A **conjunction** of two sentences $p$ and $q$ is the sentence stating "$p$ and $q$" and denoted by

$$p \wedge q$$

Truth values of conjunction can be characterized by the following table:

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

For example, the sentence

in France people speak French and $\sin \pi/2 = 1$

has the truth value 1.

An **alternative** of two sentences $p$ and $q$ is the sentence stating "$p$ or $q$" and denoted by

$$p \vee q$$

Truth values of conjunction can be characterized by the following table:

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

For example, the sentence

in France people speak French or $\sin \pi/2 = 1$

has the truth value 1.

**De Morgan's laws** establish the connection between truth values of alternative and conjunction:

$$\omega(\neg(p \wedge q)) = \omega(\neg p \vee \neg q)$$

$$\omega(\neg(p \vee q)) = \omega(\neg p \wedge \neg q)$$

An **implication** between the two sentences $p$ and $q$ is the sentence stating "if $p$ then $q$" and denoted by

$$p \to q$$

Truth values of conjunction can be characterized by the following table:

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

For example, the sentence

if in France people speak French then $\sin \pi/2 = 1$

has the truth value 1, and the sentence

if the grass is blue then $\sin \pi/2 = 1$

has the truth value 1, although the sentence

the grass is blue

has the truth value 0.

**Law of negation of implication:**

$$\omega(\neg(p \to q)) = \omega(p \wedge \neg q)$$

**Modus ponens:**

$$\omega(((p \to q) \wedge p) \to q) = 1$$

**Hypothetical syllogism:**

$$\omega(((p \to q) \wedge (q \to r)) \to (p \to r)) = 1$$

An **equivalence** between the two sentences $p$ and $q$ is the sentence stating "$p$ if and only if $q$" and denoted by

$$p \leftrightarrow q$$

Truth values of conjunction can be characterized by the following table:

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

For example, the sentence

in France people speak French if and only if $\sin \pi/2 = 1$

has the truth value 1, and the sentence

the grass is blue if and only if $\sin \pi/2 = 1$

has the truth value 0.

**First law of material equivalence:**

$$\omega((p \leftrightarrow q) \leftrightarrow ((p \to q) \wedge (q \to p))) = 1$$

**Exercises.** Decide if the following statements are true or false

(1) $(p \wedge r) \wedge (\neg q \to s) \wedge ((r \wedge s) \to \neg t) \wedge (p \wedge \neg q) \to (\neg t \wedge s)$
(2) $(\neg q \wedge s) \wedge (p \wedge \neg t) \wedge ((\neg t \wedge s) \to u) \wedge ((\neg q \wedge p) \to z) \to u \wedge z$
(3) $(p \wedge q) \wedge r \leftrightarrow p \wedge (q \wedge r)$
(4) $(p \to q) \wedge (p \to (q \to r)) \to (p \to r)$
(5) $(p \to (q \to r)) \to (q \to (p \to r))$
(6) $((p \to q) \to r) \to (p \to (q \to r))$
(7) $(p \to q) \wedge (r \to s) \to ((p \wedge r) \to (q \wedge s))$
(8) $(p \to q) \wedge (r \to s) \to ((p \wedge q) \to (r \wedge s))$
(9) $(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$
(10) $((p \vee q) \to r) \leftrightarrow ((p \to r) \wedge (q \to r))$
(11) $(p \to q) \wedge (r \to s) \to ((p \vee r) \to (q \vee s))$
(12) $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$
(13) $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$

(14) $p \wedge (q \vee r) \leftrightarrow (p \wedge r) \vee (q \wedge r)$
(15) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
(16) $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow \neg q)$
(17) $(p \rightarrow q) \wedge \neg q \rightarrow \neg p$
(18) $((p \wedge q) \rightarrow r) \leftrightarrow ((p \wedge \neg r) \rightarrow \neg q)$