Practice problems for the final

Mathematical induction and complex numbers.

(1) Solve the following systems of equations. Write your answers in the form $z = x + iy$, $w = u + iv$ where x, y, u, v are real numbers.

(a) $\begin{cases} 2(2+i)z - i(3+2i)w = 5+4i \\ (3-i)z + 2(2+i)w = 2(1+3i) \end{cases}$

(b) $\begin{cases} (4-3i)z + 2(2+i)w = 2(1+3i) \\ (2-i)z - (2+3i)w = -(1+i) \end{cases}$

(c) $\begin{cases} (2+i)z + (2-i)w = 6b - a + (2a-3b)i \\ (1-i)z + (3+i)w = a + 9b + (a+3b)i \end{cases}$

(d) $\begin{cases} \frac{z}{2-i} + \frac{w}{1+i} = 2 \\ \frac{5z}{(2-i)^2} + \frac{2w}{(1+i)^2} = 3 \end{cases}$

(2) Write the following numbers in polar form:

(a) $1, -1, i, -i,$

(b) $1+i, 1-i, -1+i, -1-i,$

(c) $1+i\sqrt{3}, 1-i\sqrt{3}, -1+i\sqrt{3}, -1-i\sqrt{3}$

(d) $\sqrt{3}+i, \sqrt{3}-i, -\sqrt{3}+i, -\sqrt{3}-i,$

(e) $\sqrt{6}+\sqrt{2}+i(\sqrt{6}-\sqrt{2}), \sqrt{6}-\sqrt{2}+i(\sqrt{6}+\sqrt{2})$

(f) $\sqrt{5}+1+i\sqrt{10-2\sqrt{5}}, \sqrt{10-2\sqrt{5}}+i(\sqrt{5}+1)$

(g) $\sqrt{5}-1+i\sqrt{10+2\sqrt{5}}, \sqrt{10+2\sqrt{5}}+i(\sqrt{5}-1)$

(h) $\sqrt{2+\sqrt{2}}+i\sqrt{2-\sqrt{2}}, \sqrt{2-\sqrt{2}}+i\sqrt{2+\sqrt{2}}$

(i) $\sqrt{2+\sqrt{3}}+i\sqrt{2-\sqrt{3}}, \sqrt{2-\sqrt{3}}+i\sqrt{2+\sqrt{3}}$

(3) Write the following numbers in polar form:

(a) i^n

(b) $(1+i)^n$

(c) $(1+i\sqrt{3})^n$

(d) $(\sqrt{6}+\sqrt{2}+i(\sqrt{6}-\sqrt{2}))^n$

(e) $(\frac{1+i}{1+i\sqrt{3}})^n$

(f) $(\frac{2-i\sqrt{6}}{1-i})^n$

(g) $[\frac{(1+i)(1+i\sqrt{3})}{\sqrt{5}+1+i\sqrt{10-2\sqrt{5}}}]^n$

(4) Use de Moivre Theorem (and mathematical induction, where applicable) to prove the following theorems:

(a) $\cos 2x = 2\cos^2 x - 1$, $\sin 2x = 2\sin x \cos x$

(b) $\cos 3x = \cos x(4\cos^2 x - 3)$, $\sin 3x = \sin x(3 - 4\sin^2 x)$

(c) $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$, $\sin 4x = 4\sin x \cos x(1 - 2\sin^2 x)$

(d) $\cos 5x = \cos x(16\cos^4 x - 20\cos^2 x + 5)$, $\sin 5x = \sin x(5 - 20\sin^2 x + 16\sin^4 x)$

(e) $\binom{\cos 2nx = \sum_{k=0}^{n}}{2n2k(-1)^k \cos^{2(n-k)} x \sin^{2k} x}$, $\binom{\sin 2nx = \sum_{k=0}^{n-1}}{2n2k+1(-1)^k \cos^{2(n-k)-1} x \sin^{2k+1} x}$

(f) $\binom{\cos(2n+1)x = \cos x \sum_{k=0}^{n}}{2n+12k(-1)^k \cos^{2(n-k)} x \sin^{2k} x}$, $\binom{\sin(2n+1)x = \sin x \sum_{k=0}^{n}}{2n+12k+1(-1)^k \cos^{2(n-k)} x \sin^{2k} x}$

(5) Use de Moivre Theorem, mathematical induction (where applicable), and the identity

$$1 + z + z^2 + \ldots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

to prove the following theorems:

(a) $\sum_{k=0}^{n} r^k \cos kx = \frac{1 - r\cos x - r^{n+1}\cos(n+1)x + r^{n+2}\cos nx}{1 - 2r\cos x + r^2}$, $\sum_{k=1}^{n} r^k \sin kx = \frac{r\sin x - r^{n+1}x + r^{n+2}\sin nx}{1 - 2r\cos x + r^2}$

(b) $\sum_{k=0}^{n} \cos(kx+y) = \frac{\cos y - r\cos(x-y) - r^{n+1}\cos[(n+1)x+y] + r^{n+2}\cos(nx+y)}{1 - 2r\cos x + r^2}$, $\sum_{k=0}^{n} r^k \sin(kx+y) = \frac{\sin y + r\sin(x-y) - r^{n+1}\sin[(n+1)x+y]+}{1 - 2r\cos x + r^2}$

(c) $\sum_{k=0}^{n} r^k \cos(k+1)x = \frac{\cos x - r - r^{n+1}\cos(n+2)x + r^{n+2}\cos(n+1)x}{1 - 2r\cos x + r^2}$, $\sum_{k=0}^{n} r^k \sin(k+1)x = \frac{\sin x - r^{n+1}\sin(n+2)x + r^{n+2}\sin(n+1)x}{1 - 2r\cos x + r^2}$

(d) $\sum_{k=0}^{n}(-1)^k r^k \cos kx = \frac{1 + r\cos x + (-1)^n r^{n+1}[\cos(n+1)x + r\cos nx]}{1 + 2r\cos x + r^2}$, $\sum_{k=0}^{n}(-1)^{k+1} r^k \sin kx = \frac{r\sin x - (-1)^n r^{n+1}[\sin(n+1)x + r\sin nx]}{1 + 2r\cos x + r^2}$

(6) Without using the polar form, find square roots of the following numbers:

(a) $i, -i$

(b) $8+6i, 8-6i, -8+6i, -8-6i$

(c) $3+4i, -3+4i, 3-4i, -3-4i$

(d) $11+60i, 11-60i, -11+60i, -11-60i$

(e) $15+8i, 15-8i, -15+8i, -15-8i$

(f) $1+i\sqrt{3}, 1-i\sqrt{3}, -1+i\sqrt{3}, -1-i\sqrt{3}$

(g) $2+3i, 2-3i, -2+3i, -2-3i$

(7) Solve the following quadratic equations:
   (a) $z^2 - 3z + 3 + i = 0$
   (b) $z^2 + (1 + 4i)z - (5 + i) = 0$
   (c) $(4 - 3i)z^2 - (2 + 11i)z - (5 + i) = 0$
   (d) $z^2 + 2(1 + i)z + 2i = 0$

Relations. Equivalence relations.

(1) Let $A = \{a, b, c, d\}$, let $R = \{(a, a), (a, b), (b, b)\}$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(2) Let $A = \{a, b, c, d\}$, let $R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a)\}$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(3) Let $A = \{a, b, c, d\}$, let $R = \{(a, b), (a, c), (b, c), (c, c), (a, a), (b, b)\}$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(4) Let $A = \{0, 1, 2\}$, let $aRb \Leftrightarrow a < b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(5) Let $A = \{1, \dots, 10\}$, let $aRb \Leftrightarrow a|b \wedge a \neq b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(6) Let $A = \{1, 2, 3, 4\}$, let $aRb \Leftrightarrow 2|a + b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(7) Let $A = \mathbb{Z}$, let $aRb \Leftrightarrow 3|a - b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(8) Let $A = \mathbb{N}$, let $aRb \Leftrightarrow 2|a + b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(9) Let $A = \mathbb{N}$, let $aRb \Leftrightarrow a \neq 0 \wedge a|b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(10) Let $A = \mathbb{N} \setminus \{0\}$, let $aRb \Leftrightarrow a|b \wedge a \neq b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(11) Let $A = \mathbb{R}$, let $aRb \Leftrightarrow a^2 = b^2$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(12) Let $A = \mathbb{R}$, let $aRb \Leftrightarrow a^2 \neq b^2$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(13) Let $A = \mathbb{C}$, let $aRb \Leftrightarrow |a| < |b|$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(14) Let $A = \mathbb{Z}$, let $aRb \Leftrightarrow |a| + |b| \neq 4$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(15) Let $A = \mathbb{R}$, let $aRb \Leftrightarrow a - b \in \mathbb{Q}$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(16) Let $A = \mathbb{Q} \times \mathbb{Q}$, let $(a, b)R(c, d) \Leftrightarrow ad = bc$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(17) Let $A =$ the set of even integers, let $aRb \Leftrightarrow 3|a - b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(18) Let $A = \mathbb{N}$, let $aRb \Leftrightarrow 2|a + b$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(19) Let $A = \mathbb{Z}$, let $aRb \Leftrightarrow 5|a - b$.Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(20) Let $A = \mathbb{Z}$, let $aRb \Leftrightarrow p|a - b$, where p is a fixed prime number. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(21) Let $A = \{1, 2, \dots, 16\}$, let $aRb \Leftrightarrow 4|a^2 - b^2$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(22) Let $A =$ polynomials in one variable t with coefficients from $\mathbb{Q}$, let $a(t)Rb(t) \Leftrightarrow \exists p, q \in \mathbb{Q} a(t) - b(t) = pt + q$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(23) Let $A = M(2, \mathbb{R})$, let $ARB \Leftrightarrow \det A = \det B$. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes
(24) Let $A =$ polynomials in one variable t with coefficients from $\mathbb{R}$, let $a(t)Rb(t) \Leftrightarrow a(t)b(t)$ is of even degree. Is R reflexive? symmetric? antisymmetric? asymmetric? transitive? linear? trichotomous? is it an equivalence relation? if so, describe its equivalence classes

Functions.

(1) For a given function f check if it is one-to-one, onto, find its inverse (if it exists), and find $f(A)$ and $f^{-1}(B)$ if:
   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -2x$, $A = B = \{1\}$

(b) $f:\mathbb{Z}\to\mathbb{Z}$, $f(x)=2x+1$, $A=\{2k:k\in\mathbb{Z}\}$, $B=\{0\}$
(c) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\cos x$, $A=[0,\frac{\pi}{4}]$, $B=[1,2]$
(d) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases}\frac{x+1}{x-1} \text{ for } x\neq 1\\ 1 \text{ for } x=1\end{cases}$, $A=[0,1]$, $B=\mathbb{N}$
(e) $f:[\frac{1}{2}\pi,\frac{3}{2}\pi]\to[-1,1]$, $f(x)=\sin x$, $A=\{\pi\}$, $B=[-1,0]$
(f) $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$, $f(n,m)=2^n\cdot(2m+1)-1$
(g) $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$, $\binom{f(n,m)=}{n+m+12+n}$
(h) $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$, $f(n,m)=\sum_{k=1}^{n+m-1}k+m-1$
(i) $f:\mathbb{Z}\times\mathbb{N}\to\mathbb{Q}$, $f(n,m)=\frac{n}{m+1}$

(2) Find the functions $f\circ g$, $g\circ f$, and their domains.
(a) $f(x)=2x^2-x$, $g(x)=3x+2$
(b) $f(x)=1-x^3$, $g(x)=1/x$
(c) $f(x)=\sin x$, $g(x)=1-\sqrt{x}$
(d) $f(x)=1-3x$, $g(x)=5x^2+3x+2$
(e) $f(x)=x+\frac{1}{x}$, $g(x)=\frac{x+1}{x+2}$
(f) $f(x)=\sqrt{2x+3}$, $g(x)=x^2+1$

Cardinality. Prove that:

(1) if $|A|=|B|$, $|C|=|D|$, and $A\cap B=C\cap D=\emptyset$, then $|A\cup B|=|C\cup D|$
(2) if $|A|=|B|$, then $|P(A)|=|P(B)|$ ($P(X)$ denotes the set of all subsets of X)
(3) $|P(A)|=|\{0,1\}^A|$ (Y^X denotes the set of all functions $X\to Y$)
(4) if $|A|=|C|$ and $|B|=|D|$, then $|A^B|=|C^D|$
(5) if $B\cap C=\emptyset$, then, for every A, $|A^{B\cup C}|=|A^B\times A^C|$
(6) $|(A^B)^C|=|A^{B\times C}|$
(7) (Cantor Theorem) $|A|<|P(A)|$
(8) (Cantor-Bernstein Theorem) if $|A|\leq|B|$ and $|B|\leq|A|$, then $|A|=|B|$.
(9) $|P(\mathbb{N})|=|\mathbb{R}|$