Math 5BI: Problem Set 8
Length of curves

Suppose that $\mathbf{C}$ is a curve in $\mathbb{R}^n$. A *parametrization* of $\mathbf{C}$ is a vector-valued function $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$ or $\mathbf{x} : [a, b] \to \mathbb{R}^n$, where $[a, b] = \{t : a \leq t \leq b\}$, such that $\mathbf{x}(t)$ traces out $\mathbf{C}$ exactly once. (In some cases we will allow the endpoints of $\mathbf{C}$ to be covered twice.) We write

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \cdot \\ x_n(t) \end{pmatrix},$$

and call the scalar-valued functions $x_1(t), \ldots, x_n(t)$ the *components* of $\mathbf{x}(t)$. A *parametrized curve* is a curve together with a parametrization of that curve.

For example, suppose that $\mathbf{C}$ is the line in R^n which passes through the point $\mathbf{p} = (p_1, \ldots, p_n)$ and is parallel to the vector $\mathbf{v} = (v_1, \ldots, v_n)$. A parametrization of $\mathbf{C}$ is

$$\mathbf{x} : \mathbb{R} \to \mathbb{R}^n, \quad \text{defined by} \quad \mathbf{x}(t) = \mathbf{p} + t\mathbf{v}.$$

If $\mathbf{C}$ is the line *segment* in $\mathbb{R}^n$ from the point $\mathbf{p} = (p_1, \ldots, p_n)$ to the point $\mathbf{q} = (q_1, \ldots, q_n)$, we we let $\mathbf{v} = \mathbf{q} - \mathbf{p}$, and parametrize $\mathbf{C}$ by

$$\mathbf{x} : [0, 1] \to \mathbb{R}^n \quad \text{where} \quad \mathbf{x}(t) = \mathbf{p} + t\mathbf{v}.$$

Problem 8.1. a. Find a parametrization for the straight line segment in $\mathbb{R}^3$ from the point $(1, 0, 3)$ to the point $(1, 1, 1)$.

b. Find a parametrized curve that might represent the trajectory of a particle in $\mathbb{R}^3$ which starts at the point $(1, 0, 3)$ when $t = 0$ and has velocity $\mathbf{v} = (3, 6, 1)$.

Suppose that we take $\mathbf{C}$ to be a circle of radius a centered at the origin and lying in a plane spanned by two unit-length vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, which are perpendicular to each other. For an arbitrary choice of t, the point

$$\mathbf{x}(t) = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2$$

lies on $\mathbf{C}$. As t ranges over the reals, $\mathbf{x}(t)$ traces out the circle infinitely many times. If we want to cover the circle just once, we can restrict t to range over the interval $[0, 2\pi]$. The circle of radius a centered at the point $\mathbf{p}$ and lying in the plane spanned by the two perpendicular unit-length vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ is parametrized by

$$\mathbf{x} : [0, 2\pi) \to R^n \quad \text{where} \quad \mathbf{x}(t) = \mathbf{p} + a\cos t\mathbf{e}_1 + a\sin t\mathbf{e}_2.$$

Problem 8.2. a. Find a parametrization for the circle of radius 3 in $\mathbb{R}^2$ with center at the point $(1, 2)$.

b. Find a parametrization for the circle of radius 3 in $\mathbb{R}^3$ with center at the origin which lies in the plane $x + y + z = 0$. (Hint: Use the cross product to find two unit-length vectors in the plane which are perpendicular to each other.

Problem 8.3. Consider the ellipse $x^2 + 2xy + 2y^2 = 1$ in $\mathbb{R}^2$. To parametrize this curve, we first write it in the form $u^2 + v^2 = 1$, where

$$\begin{aligned} u &= x + y, \\ v &= y, \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= u - v, \\ y &= v. \end{aligned}$$

Use this transformation to construct a parametrization of the ellipse.

Clearly a parametrization of a curve gives more information than just what the curve is. For example, one of Kepler's laws asserts that the planets move around the sun in ellipses. But simply knowing that a planet is moving along a given ellipse does not give a complete description of its motion. We need to know not just the curve itself, but also how the curve is parametrized, in order to know where the planet is at each point in time.

Suppose now that $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is a parametrization of a curve $\mathbf{C}$ stretching from $\mathbf{x}(a)$ to $\mathbf{x}(b)$, and that $\mathbf{x}(t)$ has smooth component functions. If $\mathbf{x}$ represents the trajectory of a moving particle and Δt is a small increment in t, the the distance traversed by the particle in the time interval Δt is approximately

$$\Delta s = |\mathbf{x}(t + \Delta t) - \mathbf{x}(t)| = \left| \frac{\mathbf{x}(t + \Delta t) - \mathbf{x}(t)}{\Delta t} \right| \Delta t.$$

In the limit as $\Delta t \to 0$, we obtain an equality of differentials,

$$ds = \left| \frac{d\mathbf{x}}{dt} \right| dt. \tag{1}$$

Here $\frac{d\mathbf{x}}{dt}$ is the derivative of the vector-valued function $\mathbf{x}(t)$ with respect to t, the differentiation being performed componentwise:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} dx_1/dt \\ \cdot \\ dx_n/dt \end{pmatrix}.$$

We can think of equation (1) as stating that the distance traversed by a moving particle in a small time interval is the speed of the particle times the length of the time interval. To find the length of the parametrized curve $\mathbf{x} : [a, b] \to R^n$, we simply integrate (1) from $t = a$ to $t = b$:

$$\text{Length of the curve } \mathbf{C} = \int_a^b \left| \frac{d\mathbf{x}}{dt} \right| dt.$$

We can write this out in terms of components as:

$$\text{Length of the curve } \mathbf{C} = \int_a^b \sqrt{(dx_1/dt)^2 + \ldots + (dx_n/dt)^2} dt.$$

Problem 8.4. Find the length of the parametrized helix

$$\mathbf{x} : [0, 2\pi] \to R^3 \quad \text{defined by} \quad \mathbf{x}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}.$$

The chain rule can be use to give a convenient formula for calculating lengths of curves which are expressed in polar coordinates. Recall that polar coordinates r, θ are related to the usual coordinates x, y by the equations

$$x = r\cos\theta, \quad y = r\sin\theta.$$

If r and θ are functions of t, then

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \begin{pmatrix} \frac{dr}{dt} \\ \frac{d\theta}{dt} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \frac{dr}{dt} \\ \frac{d\theta}{dt} \end{pmatrix}.$$

Problem 8.5. Show that

$$(dx/dt)^2 + (dy/dt)^2 = (dr/dt)^2 + r^2(d\theta/dt)^2,$$

which implies the following formula for arc length in polar coordinates:

$$\text{Length of the curve } \mathbf{C} = \int_a^b \sqrt{(dr/dt)^2 + r^2(d\theta/dt)^2}\, dt.$$

Problem 8.6. Find the length of the spiral, defined in polar coordinates by the formula,

$$r = e^\theta, \quad 0 \le \theta \le 2\pi.$$

You are not responsible for the remaining material in this work sheet, but I hope some of you will find it interesting!

Special relativity provides a striking application of a slightly altered arc length integral. Here are two quotations from the creators of the theory:

> I am convinced that the philosophers have had a harmful effect upon the progress of scientific thinking in removing certain fundamental concepts from the domain of empiricism, where they are under our control, to the intangible heights of the *a priori*. For even if it should appear that the universe of ideas cannot be deduced from experience by logical means, but is, in a sense, a creation of the human mind, without which no science is possible, nevertheless this universe of ideas is just as little independent of the nature of our experiences as clothes are of the form of the human body. This is particularly true of our concepts of time and space, which physicists have been obliged by the facts to bring down from the Olympus of the *a priori* in order to adjust them and put them in a serviceable condition. (Albert Einstein, *The meaning of relativity*, Princeton Univ. Press, Princeton, New Jersey, 1955, p.2.)

> Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality. (Hermann Minkowski, *Space and time*, in *The principle of relativity*, Dover, New York, 1923, p.75.)

Special relativity teaches that the time measured by a clock depends upon its motion. The arena for special relativity is space-time, which is given space coordinates (x, y, z) and a time coordinate t. One imagines that the points of space-time locate *events* which occur at a specific location and at a specific time—for example, the exploding of a firecracker at a given location (x, y, z) and at a given time t would be regarded as an event.

An individual living within space-time witnesses a continuum of events, represented by a curve in space-time called his world line. The curve starts at an event, the birth of the individual, located at a point (t_0, x_0, y_0, z_0), and ends at another event, the death of the individual, located at a point (t_1, x_1, y_1, z_1). Such a curve is called a *world line*.

One of the tenets of special relativity is that nothing can move faster than the speed of light. Thus, if c denotes the speed of light,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \leq c.$$

If we square both sides, and multiply by dt^2, this inequality becomes

$$c^2dt^2 - dx^2 - dy^2 - dz^2 \geq 0.$$

The subjective time measured by the individual is measured by the "element of arc length,"

$$ds^2 = dt^2 - \frac{dx^2}{c^2} - \frac{dy^2}{c^2} - \frac{dz^2}{c^2} = \frac{c^2dt^2 - dx^2 - dy^2 - dz^2}{c^2}.$$

Suppose that $\mathbf{C}$ is an individual's world line, and let us parametrize it with a parameter τ, say

$$\mathbf{x}(\tau) = \begin{pmatrix} t(\tau) \\ x(\tau) \\ y(\tau) \\ z(\tau) \end{pmatrix}, \quad a \leq \tau \leq b.$$

Then the time measured by the individual between the two events $x(\tau_0)$ and $x(\tau_1)$ is given by

$$\begin{aligned}\int_{\mathbf{C}} ds &= \int_{\mathbf{C}} \frac{1}{c}\sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2} \\ &= \int_{\tau_0}^{\tau_1} \frac{1}{c}\sqrt{c^2(dt/d\tau)^2 - (dx/d\tau)^2 - (dy/d\tau)^2 - (dz/d\tau)^2}d\tau.\end{aligned}$$

Just like in the case of arc length, this integral depends only on the world line $\mathbf{C}$, not on the choice of parameter τ.

Extra Problem 8.7. If an observer is not moving, $dx/d\tau = dy/d\tau = dz/d\tau = 0$. Show that in this case, the time measured by a stationary clock is given by the t-coordinate.

Extra Problem 8.8. Suppose that $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$, where $\mathbf{C}_1$ is the straight line segment from from the event $(0,0,0,0)$ to the event $(T, 0, 0, .99cT)$ and $\mathbf{C}_2$ is the straight line segment from from the event $(T, 0, 0, .99cT)$ to the event $(2T, 0, 0, 0)$. Find the time that is measured by an observer that moves along the world line $\mathbf{C}$. Show that this is smaller than the time measure by an observer which moves along the straight line segment from $(0,0,0,0)$ to $(2T, 0, 0, 0)$.

We remark that this is known as the **twin paradox**.