Math 5BI: Problem Set 7
Constrained maxima and minima

One of the most useful applications of differential calculus of several variables is to the problem of finding maxima and minima of functions of several variables. The method of *Lagrange multipliers* finds maxima and minima of functions of several variables which are subject to constraints.

To understand the method in the simplest case, let use suppose that S is a surface in R^3 defined by the equation $\phi(x, y, z) = 0$, where ϕ is a smooth real-valued function of three variables. Let us assume, for the time being, that S has no boundary and that the gradient $\nabla\phi$ is not zero at any point of S, so that the tangent plane to S at a given point (x_0, y_0, z_0) is given by the equation

$$(\nabla\phi)(x_0, y_0, z_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0.$$

Problem 7.1. Find an equation for the plane tangent to the surface

$$x^2 + 4y^2 + 9z^2 = 14$$

at the point $(1, 1, 1)$.

Problem 7.2. a. Sketch the level sets of the function $f(x, y) = x^2 + y^2$.

b. Sketch the curve $4x^2 + y^2 = 4$.

c. At which points of the curve $4x^2 + y^2 = 4$ does the function f assume its maximum and minimum values?

Let S be a surface in R^3 defined by the equation $\phi(x, y, z) = 0$ and let $f(x, y, z)$ is a smooth function of three variables. Suppose that we are interested in finding the maximum and minimum values of f on the surface S. Thus we seek to maximize or minimize f subject to the constraint $\phi(x, y, z) = 0$.

The function f determines a family of level surfaces $f(x, y, z) = c$. If (x_0, y_0, z_0) is a point on S at which a maximum or minimum value is attained, then S will be tangent to the level set

$$f(x, y, z) = c, \quad \text{where} \quad c = f(x_0, y_0, z_0).$$

Thus $(\nabla f)(x_0, y_0, z_0)$ must be a multiple of $(\nabla\phi)(x_0, y_0, z_0)$, i.e. there must exist a constant λ_0 such that

$$(\nabla f)(x_0, y_0, z_0) = \lambda_0(\nabla\phi)(x_0, y_0, z_0).$$

In other words, $(x_0, y_0, z_0, \lambda_0)$ must be a solution to the system

$$(\nabla f)(x, y, z) = \lambda(\nabla \phi)(x, y, z), \quad \phi(x, y, z) = 0, \tag{1}$$

or equivalently,

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial \phi}{\partial x}, \quad \frac{\partial f}{\partial y} = \lambda \frac{\partial \phi}{\partial y}, \quad \frac{\partial f}{\partial z} = \lambda \frac{\partial \phi}{\partial z}, \quad \phi = 0. \tag{2}$$

In order to remember these systems more easily, we make use of the function

$$H(x, y, z, \lambda) = f(x, y, z) - \lambda \phi(x, y, z).$$

The solutions to either (1) or (2) are just the critical points of H.

If $(x_0, y_0, z_0, \lambda_0)$ is a critical point for H, we say that (x_0, y_0, z_0) is a *critical point* for f on the surface S defined by the equation $\phi = 0$. These critical points are the candidates for the points on S at which f assumes its maximum and minimum values.

This process for finding the candidates for maxima and minima is called the *method of Lagrange multipliers*.

A theorem from analysis says that a continuous function always assumes its maximum and minimum values on any closed bounded surface in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Problem 7.3. Find the maximum and minimum values of the function $f(x, y, z) = 4x + 2y + 6z$ on the ellipsoid S defined by the equation $x^2 + 2y^2 + 3z^2 - 1 = 0$.

Problem 7.4. a. Suppose that we want to find the maximum and minimum values for the function $f(x, y, z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 1$. In this case, the constraint equation is

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0,$$

and the function H is given by the formula

$$H(x, y, z, \lambda) = xyz - \lambda(x^2 + y^2 + z^2 - 1).$$

Find the critical points of H.

b. Use the critical points you found in part a to find the maximum and minimum values for $f(x, y, z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 1$.

Problem 7.5. Here is a more "practical" application of Lagrange multipliers. Suppose that we want to construct a box, open on the top, of given volume, say 32 cubic inches, utilizing a minimal amount of cardboard. Let

$$x = \text{width of box}, \quad y = \text{length of box}, \quad z = \text{height of box}.$$

Then the area of the base is xy, while the area of the four sides is $2xz + 2yz$, so the total area of cardboard used is

$$f(x, y, z) = xy + 2xz + 2yz.$$

We need to minimize this function f subject to the constraint

$$\text{volume of the box} = xyz = 32.$$

Our constraint function in this case can be taken to be

$$\phi(x, y, z) = xyz - 32.$$

Find the dimensions of the box which uses the least amount of cardboard.