Math 5BI: Problem Set 4
Functions of many variables

Suppose that

$$z = f(x_1, \ldots, x_n), \quad \text{and} \quad x_1 = x_1(t), \ldots, x_n = x_n(t),$$

where the functions $f, x_1, \ldots, x_n$ are continuously differentiable. Then the chain rule states that

$$\frac{dz}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt},$$

or as is sometimes written,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial z}{\partial x_n}\frac{dx_n}{dt}. \tag{1}$$

We can define the *gradient* of $f(x_1, \ldots, x_n)$ at the point $(c_1, \ldots, c_n) \in \mathbb{R}^n$ to be

$$\nabla f(c_1, \ldots, c_n) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(c_1, \ldots, c_n) \\ \cdot \\ \frac{\partial f}{\partial x_n}(c_1, \ldots, c_n) \end{pmatrix}.$$

Then the chain rule can be restated in vector form as

$$z'(t_0) = \nabla f(\mathbf{x}(t_0)) \cdot \mathbf{x}'(t_0),$$

where

$$\mathbf{x}(t_0) = \begin{pmatrix} x_1(t_0) \\ \cdot \\ x_n(t_0) \end{pmatrix} \quad \text{and} \quad \mathbf{x}'(t_0) = \begin{pmatrix} x_1'(t_0) \\ \cdot \\ x_n'(t_0) \end{pmatrix}.$$

Just as in the two and three-dimensional cases, the gradient points in the direction of maximal increase of f and its magnitude is the rate of change of f with respect to time for a particle moving at unit speed in the direction of the gradient.

The linearization of $f(x_1, \ldots, x_n)$ at the point $\mathbf{c} = (c_1, \ldots, c_n)$ is the function

$$L(x_1, \ldots, x_n) = f(c_1, \ldots, c_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(c_1, \ldots, c_n)(x_i - c_i),$$

or in vector notation

$$L(\mathbf{x}) = f_i(\mathbf{c}) + \nabla f(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{c}).$$

Definition. A point $(c_1, \ldots, c_n) \in \mathbb{R}^n$ is said to be a *critical point* for a continuously differentiable function $f(x_1, \ldots, x_n)$ if $\nabla f(c_1, \ldots, c_n) = 0$.

If $(c_1, \ldots, c_n)$ is a point at which $f(x_1, \ldots, x_n)$ assumes a maximum or minimum value, then $(c_1, \ldots, c_n)$ must be a critical point for $f(x_1, \ldots, x_n)$.

Problem 4.1. a. What are the critical points of the function

$$f(x_1, x_2, x_3, x_4) = x_1^2 + 2x_1x_2 - x_2^2 + x_3^2 + x_4^2 - 2x_1 + 4x_3?$$

b. Suppose that you know that the function

$$f(x_1, x_2, x_3, x_4) = x_1^2 + 2x_1x_2 - x_2^2 + x_3^2 + x_4^2 - 2x_1 + 4x_3$$

assumes a minimum at some point. What is that point?

A *mapping* $\mathbf{F}$ from R^n to R^m is defined by m functions of n variables:

$$\begin{aligned} y_1 &= f_1(x_1, \ldots x_n), \\ y_2 &= f_2(x_1, \ldots x_n), \\ \ldots & \\ y_m &= f_m(x_1, \ldots x_n). \end{aligned} \tag{2}$$

It is often convenient to write this in vector notation. If we set

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \cdot \\ y_m \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \cdot \\ x_n \end{pmatrix}, \quad \text{and} \quad \mathbf{F}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \cdot \\ f_m(\mathbf{x}) \end{pmatrix},$$

then we can write

$$\mathbf{y} = \mathbf{F}(\mathbf{x}).$$

Important examples of mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the affine transformations,

$$\begin{aligned} y_1 &= a_{11}x_1 + \ldots + a_{1n}x_n + b_1, \\ y_2 &= a_{21}x_1 + \ldots + a_{2n}x_n + b_2, \\ \ldots & \\ y_m &= a_{m1}x_1 + \ldots + a_{mn}x_n + b_m. \end{aligned}$$

If all of the b_i's are zero, such transformations are said to be linear.

If $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is a nonlinear mapping, it is sometimes convenient to find its best affine approximation near a given point $(c_1, \ldots, c_n)$. This is done by finding the best affine approximation of each of its component functions f_i. Thus the best affine approximation to

$$\begin{aligned} y_1 &= f_1(x_1, \ldots, x_n), \\ y_2 &= f_2(x_1, \ldots, x_n), \\ \ldots & \\ y_m &= f_m(x_1, \ldots, x_n) \end{aligned}$$

is

$$\begin{array}{rcl} y_1 & = & L_1(x_1, \ldots, x_n), \\ y_2 & = & L_2(x_1, \ldots, x_n), \\ \cdots & & \\ y_m & = & L_m(x_1, \ldots, x_n), \end{array}$$

where

$$L_i(x_1, \ldots, x_m) = L_i(\mathbf{x}) = f_i(\mathbf{c}) + \nabla f_i(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{c}).$$

We can write this last set of equations in matrix form as

$$\begin{pmatrix} y_1 \\ \cdot \\ y_m \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{c}) \\ \cdot \\ f_m(\mathbf{c}) \end{pmatrix} + \begin{pmatrix} \partial f_1/\partial x_1(\mathbf{c}) & \cdot & \partial f_1/\partial x_n(\mathbf{c}) \\ \cdot & \cdot & \cdot \\ \partial f_m/\partial x_1(\mathbf{c}) & \cdot & \partial f_m/\partial x_n(\mathbf{c}) \end{pmatrix} \begin{pmatrix} x_1 - c_1 \\ \cdot \\ x_n - c_n \end{pmatrix},$$

or more succinctly as

$$\mathbf{y} = \mathbf{F}(\mathbf{c}) + \mathbf{DF}(\mathbf{c})(\mathbf{x} - \mathbf{c}),$$

where

$$\mathbf{DF}(\mathbf{c}) = \begin{pmatrix} \partial f_1/\partial x_1(\mathbf{c}) & \cdot & \partial f_1/\partial x_n(\mathbf{c}) \\ \cdot & \cdot & \cdot \\ \partial f_m/\partial x_1(\mathbf{c}) & \cdot & \partial f_m/\partial x_n(\mathbf{c}) \end{pmatrix}.$$

The matrix $\mathbf{DF}(\mathbf{c})$ is called the *derivative* of the mapping $\mathbf{F}$ at $\mathbf{c}$. The affine mapping

$$\mathbf{L}(\mathbf{x}) = \mathbf{F}(\mathbf{c}) + \mathbf{DF}(\mathbf{c})(\mathbf{x} - \mathbf{c})$$

is called the *linearization* of $\mathbf{F}$ at $\mathbf{c}$.

Problem 4.2. a. Find the derivative of the mapping

$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ xy \end{pmatrix}$$

at the point $(1, 2)$.

b. Find the linearization of the mapping

$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ xy \end{pmatrix}$$

at the point $(2, 3)$.

For example, suppose that

$$\begin{array}{rcl} dx_1/dt & = & f_1(x_1, \ldots x_n), \\ dx_2/dt & = & f_2(x_1, \ldots x_n), \\ \cdots & & \\ dx_m/dt & = & f_m(x_1, \ldots x_n). \end{array} \tag{3}$$

is a **nonlinear system** of linear differential equations, written in vector form as

$$d\mathbf{x}/dt = \mathbf{F}(\mathbf{x}).$$

It is often quite difficult to find the explicit solutions to such a system. We say that $\mathbf{x} = \mathbf{c}$ is a constant solution to this system if $\mathbf{F}(\mathbf{c}) = \mathbf{0}$. In this case, we can linearize the system (**??**) near $\mathbf{c}$, by replacing the right-hand side by its linearization

$$d\mathbf{x}/dt = \mathbf{F}'(\mathbf{c})(\mathbf{x} - \mathbf{c}).$$

In terms of components, the new linearized system is

$$\begin{aligned} dx_1/dt &= (\partial f_1/\partial x_1)(\mathbf{c})(x_1 - c_1) + \ldots + (\partial f_1/\partial x_n)(\mathbf{c})(x_n - c_n), \\ dx_2/dt &= (\partial f_2/\partial x_1)(\mathbf{c})(x_1 - c_1) + \ldots + (\partial f_2/\partial x_n)(\mathbf{c})(x_n - c_n), \\ \ldots & \\ dx_m/dt &= (\partial f_n/\partial x_1)(\mathbf{c})(x_1 - c_1) + \ldots + (\partial f_n/\partial x_n)(\mathbf{c})(x_n - c_n). \end{aligned}$$

The solutions to this **linear system** are far easier to determine (for example, we can use techniques of Math 5AI), and they will approximate the solutions to the original nonlinear system near $\mathbf{c}$.

Problem 4.3. a. Consider the Volterra-Lotka predator-prey system of differential equations

$$\frac{dx}{dt} = x(1 - y),$$
$$\frac{dy}{dt} = y(x - 1).$$

Here $x(t)$ represents the number of millions of rabbits on an island and $y(t)$ represents the number of thousands of foxes. This system can be written in vector form as

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}), \quad \text{where} \quad \mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(1-y) \\ y(x-1) \end{pmatrix}.$$

Find the linearization of $\mathbf{F}$ at the equilibrium point $(1, 1)$.

b. Find the linearization of the predator-prey system of differential equations at the point $(1, 1)$.

c. Consider the related system of differential equations

$$\frac{dx}{dt} = x(1 - y) - \frac{1}{10}x^2,$$
$$\frac{dy}{dt} = y(x - 1).$$

Find the linearization of this system at the point $(1, 9/10)$. Is the linearization stable? What does this suggest about the stability of the equilibrium $(1, 9/10)$ for the nonlinear system?