Math 5BI: Problem Set 3
Functions of three variables

The notion of linearization and the chain rule can be extended to functions of n variables, where n can be arbitrary. In this problem set, we want to consider the case $n = 3$.

Suppose that $f(x, y, z)$ is a function of three variables. When it exists, the partial derivative of $f(x, y, z)$ with respect to x at (x_0, y_0, z_0) is given by the formula

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0, z_0) - f(x_0, y_0, z_0)}{h}.$$

(Notice the use of limits in the definition.) The partial derivatives

$$\frac{\partial f}{\partial y}(x_0, y_0, z_0) \quad \text{and} \quad \frac{\partial f}{\partial z}(x_0, y_0, z_0)$$

are defined by similar formulae, which you could easily write down.

We say that the function $f(x, y, z)$ is *continuously differentiable* or *smooth* if it has partial derivatives at every point, and the functions

$$\frac{\partial f}{\partial x}(x, y, z), \quad \frac{\partial f}{\partial y}(x, y, z) \quad \text{and} \quad \frac{\partial f}{\partial z}(x, y, z)$$

are continuous.

The *gradient* of a continuously differentiable function $f(x, y, z)$ at the point (x_0, y_0, z_0) is the vector

$$\nabla f(x_0, y_0, z_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0, z_0) \\ \frac{\partial f}{\partial y}(x_0, y_0, z_0) \\ \frac{\partial f}{\partial z}(x_0, y_0, z_0) \end{pmatrix}.$$

If we let (x, y, z) vary, we get a function

$$(x, y, z) \in \mathbb{R}^3 \mapsto (\nabla f)(x, y, z) \in \mathbb{R}^3$$

which is just called the *gradient* of f. More generally, a function

$$\mathbf{X} : \mathbb{R}^3 \to \mathbb{R}^3$$

is often called a *vector field*. Thus the gradient ∇f of f is an example of a vector field.

Problem 3.1. a. We can sketch the vector field ∇f by drawing an arrow at each point (x, y, z) of $\mathbb{R}^3$ in the direction of $\nabla f(x, y, z)$. Sketch ∇f, when

$$f(x, y, z) = \frac{1}{4}x^2 + \frac{1}{4}y^2 + \frac{1}{4}z^2.$$

b. Newton's expression for gravitational potential caused by a body of mass M at the origin of $\mathbb{R}^3$ was

$$f(x, y, z) = -\frac{GM}{\sqrt{x^2 + y^2 + z^2}},$$

where G is a universal constant. What is

$$\mathbf{F}(x, y, z) = -\nabla f(x, y, z)$$

in this case. Newton called $\mathbf{F}$ the gravitational force which acts on a point body of mass one.

c. Sketch the vector field $\mathbf{F}(x, y, z)$ of part b in the case where $GM = (1/4)$.

Just as in the case of two variables, the linearization of $f(x, y, z)$ at (x_0, y_0, z_0) is the function

$$L(x, y, z) = f(x_0, y_0, z_0) + \frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0).$$

Just as in the case of functions of two variables, we expect the linearization to be very close approximation to the function near the point (x_0, y_0, z_0).

Problem 3.2. a. If $f(x, y, z) = x^2 + y^2 + z^2$, what is its linearization at the point $(3, 1, 2)$?

b. Use the linearization to find a close approximation to f at the point

$$(3.001, .999, 2.002).$$

Suppose now that we have a smooth parametrized curve in $\mathbb{R}^3$,

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

This might represent the position of a moving particle in $\mathbb{R}^3$ at time t. The velocity of the particle at time t would then be

$$\mathbf{x}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}.$$

We can take the composition of the functions f and $\mathbf{x}(t)$ obtaining a new function

$$h(t) = (f \circ \mathbf{x})(t) = f(x(t), y(t), z(t)).$$

In this context, the chain rule states

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}. \tag{1}$$

The chain rule can be restated in vector form as

$$h'(t_0) = \nabla f(\mathbf{x}(t_0)) \cdot \mathbf{x}'(t_0),$$

where

$$\mathbf{x}(t_0) = \begin{pmatrix} x(t_0) \\ y(t_0) \\ z(t_0) \end{pmatrix} \quad \text{and} \quad \mathbf{x}'(t_0) = \begin{pmatrix} x'(t_0) \\ y'(t_0) \\ z'(t_0) \end{pmatrix},$$

for any choice of t_0.

Problem 3.3. The *level set* of a continuously differentiable function $f(x, y, z)$ is a set of points (x, y, z) which satisfy the equation $f(x, y, z) = c$. Sketch the level sets of the function $f(x, y, z) = x^2 + y^2 + z^2$.

Problem 3.4. a. Suppose that the level set $f(x, y, z) = c$ is a smooth surface S. Use the chain rule to show that if this is the case, then ∇f is perpendicular to S at any point of S.

b. Find a vector perpendicular to the surface $x^2 + y^2 - z^2 = 1$ at the point $(1, 1, 1)$.

c. Find an equation for the plane tangent to the surface $x^2 + y^2 - z^2 = 1$ at the point $(1, 1, 1)$.