Math 5BI: Problem Set 2
The chain rule

Problem 2.1. Part I. You have shown that if $f(x, y)$ is a continuously differentiable function of two variables then its local linearization at (x_0, y_0) is given by

$$L(x,y) = f(x_0, y_0) + \left(\frac{\partial f}{\partial x}\right)(x_0, y_0)(x - x_0) + \left(\frac{\partial f}{\partial y}\right)(x_0, y_0)(y - y_0)$$

Consider the function $M : \mathbb{R} \to \mathbb{R}^2$ defined by $M(t) = (x_0, y_0) + (2t, 3t)$. Find the composite function, $L(M(t)) : \mathbb{R} \to \mathbb{R}$. This should be the local linearization of $f(M(t))$. Discuss this.

Suppose that $\gamma(t) = (x(t), y(t), z(t))$ is a differentiable parametrized curve in $\mathbb{R}^3$ which lies on the surface S defined by the equation $z = f(x, y)$, where f is a continuously differentiable function of two variables. Thus

$$z(t) = f(x(t), y(t)). \tag{1}$$

It is intuitively clear, and can be proven rigorously, that the velocity vector $\gamma'(t_0)$ is tangent to the surface S at $\gamma(t_0)$, for any choice of t_0.

Remark. Since there are only 26 letters in the alphabet, scientists often run out of distinct letters to represent variables and functions and so on. Thus one writes $x = x(t)$, where the first x represents the *variable* x and the second x stands for a *function* of t. This is one way of economizing on use of letters. Another way of increasing the number of available letters is to use the Greek alphabet, including for example, the Greek letter gamma, γ.

Problem 2.1. Part II.Use Part I of 2.1 to explain why

$$z'(t_0) = \left(\frac{\partial f}{\partial x}\right)(x(t_0), y(t_0))x'(t_0) + \left(\frac{\partial f}{\partial y}\right)(x(t_0), y(t_0))y'(t_0), \tag{2}$$

where the prime denotes the derivative with respect to t.

The formula (2) is called the *chain rule*. It can be written in many different forms with various choices of notation. If we leave out the constants t_0, x_0, and y_0, we can simplify (2) to

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{3}$$

If we use the notation $\partial z/\partial x$ for $\partial f/\partial x$ and $\partial z/\partial y$ for $\partial f/\partial y$, we can write the chain rule in the form

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

In this notation, we call t the *independent variable*, z the *dependent variable*, and x and y the *intermediate variables*. All of these versions of the chain rule appear in books on physics and engineering.

Problem 2.2. Use the chain rule to answer the following question: The volume of a cylindrical can of radius r and height h is $V = \pi r^2 h$. If at a certain time

$$r = 2 \text{ ft}, \quad h = 5 \text{ ft}, \quad \frac{dr}{dt} = 1 \text{ ft/sec}, \quad \frac{dh}{dt} = 2 \text{ ft/sec},$$

what is dV/dt?

The parametrized curve $\gamma(t)$ utilized in Problem 2.1 projects to a parametrized curve $\mathbf{x}(t) = (x(t), y(t))$ in the (x, y)-plane. Conversely given any smooth curve $\mathbf{x}(t) = (x(t), y(t))$ in the (x, y)-plane we have a corresponding curve $\gamma(t)$ on the surface S defined by the equation $z = f(x, y)$, namely

$$\gamma(t) = (x(t), y(t), z(t)), \quad \text{where} \quad z(t) = f(x(t), y(t)).$$

Definition. The *gradient* of a continuously differentiable function $f(x, y)$ at the point (x_0, y_0) is

$$\nabla f(x_0, y_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) \\ \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix}.$$

Problem 2.3. a. Suppose that

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

is a parametrized curve in the (x, y)-plane, representing the trajectory of a moving particle in the plane, the variable t representing time. Note that the velocity of the particle at time t_0 is just

$$\mathbf{v}(t_0) = \mathbf{x}'(t_0) = \begin{pmatrix} x'(t_0) \\ y'(t_0) \end{pmatrix}.$$

Use (2) to show that

$$z'(t_0) = \nabla f(x(t_0), y(t_0)) \cdot \mathbf{x}'(t_0), \quad \text{where} \quad \mathbf{x}'(t_0) = \begin{pmatrix} x'(t_0) \\ y'(t_0) \end{pmatrix}.$$

b. Suppose that $f(x, y)$ represents the temperature at the point (x, y). Show that

(rate of change of temperature with respect to t) = (gradient of f) $\cdot$ (velocity).

c. Let $\mathbf{u} = (u_1, u_2)$ be a unit-length vector (so $\mathbf{u} \cdot \mathbf{u} = 1$) and let (x_0, y_0) be a point in the plane $\mathbb{R}^2$. Show that the speed of the parametrized curve

$$\mathbf{x}(t) = (x_0, y_0) + t\mathbf{u}$$

is one. Thus $\mathbf{x}(t)$ has unit speed and starts at (x_0, y_0) at time $t = 0$.

d. We can think of a unit-length vector $\mathbf{u} = (u_1, u_2)$ as defining a direction in $\mathbb{R}^2$. What is the rate of change of $z = f(x, y)$ with respect to time t in the direction of $\mathbf{u}$ at the point (x_0, y_0)?

e. Show that the gradient of f at (x_0, y_0) points in the direction of maximum increase of f, and its magnitude is the rate of change of f in this direction, if one moves from (x_0, y_0) with unit speed.

Problem 2.4. In this problem you consider curves that are defined *implicitly.* Use the idea that the gradient points in direction of maximal increase, which is perpendicular to the tangent direction along a level set of a function $z = f(x, y)$.

a. Find a nonzero vector which is perpendicular to the curve $x^4 + y^4 = 17$ at the point $(1, 2)$.

b. Find an equation for the line which is perpendicular to the curve $x^4 + y^4 = 17$ at the point $(1, 2)$.

c. Find an equation for the line which is tangent to the curve $x^4 + y^4 = 17$ at the point $(1, 2)$.