Math 5BI: Problem Set 12
Green's Theorem

If $\mathbf{F}(x, y) = M(x, y)\mathbf{i}$ is a vector field on a region D in the plane, where $M(x, y)$ and $N(x, y)$ are smooth functions on D, and $d\mathbf{x} = dx\mathbf{i} + dy\mathbf{j}$, then

$$\mathbf{F}(x, y) \cdot d\mathbf{x} = M(x, y)dx + N(x, y)dy$$

is called a *differential*.

In particular if $\mathbf{F} = \nabla f$, where $f(x, y)$ is a smooth scalar-valued function on D then

$$\mathbf{F}(x, y) \cdot d\mathbf{x} = \frac{\partial f}{\partial x}(x, y)dx + \frac{\partial f}{\partial y}(x, y)dy.$$

We write

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

We say that a differential $Mdx + Ndy$ is *exact* if $Mdx + Ndy = df$ for some smooth function f. Note that if $Mdx + Ndy$ is exact, then

$$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y} \qquad \Rightarrow \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial M}{\partial y}.$$

Problem 12.1. Determine which of the following differentials are exact:

$$xdy - ydx, \quad ydx + xdy, \quad e^y dx + xe^y dy.$$

Problem 12.2. a. Write the differential equation

$$\frac{dy}{dx} = -\frac{2xy + e^y}{x^2 + xe^y} \quad \text{in the form} \quad Mdx + Ndy = 0. \tag{1}$$

b. Is it true that

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}?$$

c. Find a function $f(x, y)$ such that $df = Mdx + Ndy$. Then $f(x, y) = c$, where c is an arbitrary constant, is the general solution to the differential equation (**??**).

Remark. This method of solving ordinary differential equations is called the *method of exact differentials*.

A differential $Mdx + Ndy$ on a region D in the plane is said to be *closed* if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Note that every exact differential on a region D in the plane is closed, but we will see that there are closed differentials on some regions D which are closed but not exact!

Green's Theorem relates double integrals to line integrals:

Green's Theorem. *Let D be a bounded region in the (x, y)-plane, bounded by a piecewise smooth curve ∂D, directed so that as it is traversed in the positive direction, the region D lies on the left. Let $M(x, y)dx + N(x, y)dy$ be a differential on $D \cup \partial D$ whose component functions M and N are smooth on $D \cup \partial D$. Then*

$$\int_{\partial D} Mdx + Ndy = \int\int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

Problem 12.3. Suppose that $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Verify that Green's theorem holds for D and the differential $Mdx + Ndy = ydx - xdy$.

To prove Green's Theorem, it suffices to prove the two simpler formulae

$$\int_{\partial D} Mdx = \int\int_D \left(-\frac{\partial M}{\partial y} \right) dxdy \tag{2}$$

and

$$\int_{\partial D} Ndy = \int\int_D \left(\frac{\partial N}{\partial x} \right) dxdy. \tag{3}$$

We focus on (**??**); the proof of (**??**) is similar.

To prove (**??**) in the case where D is of the special form

$$D = \{(x, y) \in R^2 : a \leq x \leq b, \phi(x) \leq y \leq \psi(x)\},$$

of type I in the terminology we used before, we note that the boundary curve ∂D divides up into four pieces:

$$\partial D = \mathbf{C}_1 + \mathbf{C}_3 - \mathbf{C}_2 - \mathbf{C}_4,$$

which have the following parametrizations:

$$\mathbf{C}_1 : x = t, y = \phi(t), a \leq x \leq b,$$

$$\mathbf{C}_2 : x = t, y = \psi(t), a \leq x \leq b,$$

$$\mathbf{C}_3 : x = a, y = t, \phi(a) \leq x \leq \psi(a),$$

$$\mathbf{C}_4 : x = b, y = t, \phi(b) \leq x \leq \psi(b).$$

Problem 12.4. a. Show that $dx = 0$ along $\mathbf{C}_3$ and $\mathbf{C}_4$. Use this fact to evaluate

$$\int_{\mathbf{C}_3} M dx \quad \text{and} \quad \int_{\mathbf{C}_4} M dx.$$

b. Show that

$$\int_{\partial D} M dx = \int_{\mathbf{C}_1} M dx - \int_{\mathbf{C}_2} M dx.$$

c. Show that

$$\int_{\partial D} M dx = \int_a^b M(t, \phi(t)) dt - \int_a^b M(t, \psi(t)) dt = \int_a^b [M(x, \phi(x)) - M(x, \psi(x))] dx.$$

d. Use the fundamental theorem of calculus to show that

$$\int_{\partial D} M dx = \int_a^b \int_{\phi(x)}^{\psi(x)} \left[-\frac{\partial M}{\partial y}(x, y) \right] dx dy = -\int \int_D \frac{\partial M}{\partial y}(x, y) dx dy.$$

This establishes (**??**) in the case where D is of type I.

The general case of (**??**) is obtained by dividing a given region D into a disjoint union of regions D_i of type I. In this case,

$$\int \int_D -\frac{\partial M}{\partial y} dx dy = \sum \int \int_{D_i} -\frac{\partial M}{\partial y} dx dy = \sum \int_{D_i} M dx = \int_{\partial D} M dx,$$

because the parts of the boundaries of the D_i's which lie inside D cancel in pairs.

Problem 12.5. a. Suppose that

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} - \{(0, 0)\}.$$

Show that the differential

$$M dx + N dy = \frac{y dx - x dy}{x^2 + y^2} \tag{4}$$

is closed.

b. Let $\mathbf{C}$ be the circle $x^2 + y^2 = 1$ directed once in the counterclockwise direction. Evaluate the line integral

$$\int_{\mathbf{C}} \frac{y dx - x dy}{x^2 + y^2}.$$

c. Does your calculation in part c show that the differential (**??**) is closed but not exact? Why or why not?

A region $D \subset \mathbb{R}^2$ is said to be *convex* if

$$p \in D \quad \text{and} \quad q \in D \qquad \Rightarrow \qquad (1 - t)p + tq \in D \quad \text{for all } t \in [0, 1].$$

Problem 12.6. Is the region

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} - \{(0, 0)\}$$

convex? Why or why not?

Poincaré Lemma. *If D is a* **convex** *region in $\mathbb{R}^2$ then every closed differential on D is exact.*

Problem 12.7. a. Suppose that $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a smooth vector field on a region D in the (x, y)-plane, bounded by a piecewise smooth curve ∂D, directed so that as it is traversed in the positive direction, the region D lies on the left. Let $\mathbf{T}$ denote the unit-length tangent vector to ∂D and let $\mathbf{N}$ denote the outward pointing unit-length normal to ∂D. Show that

$$(P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}) \cdot \mathbf{N} = (-Q(x, y)\mathbf{i} + P(x, y)\mathbf{j}) \cdot \mathbf{T}$$

along ∂D.

b. Use Green's Theorem to prove the Divergence Theorem:

Divergence Theorem. *Let D be a bounded region in the (x, y)-plane, bounded by a piecewise smooth curve ∂D. Let $\mathbf{F}(x, y) = P(x, y)dx + Q(x, y)dy$ be a differential on $D \cup \partial D$ whose component functions P and Q are smooth. Then*

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{N} ds = \int\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dxdy.$$

Problem 12.8. Use the Divergence Theorem to evaluate the line integral

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{N} ds,$$

where $\mathbf{C}$ is the unit circle $x^2 + y^2 = 1$ and

$$\mathbf{F} = (y \cos e^y)\mathbf{i} + (x + y)\mathbf{j}.$$