Math 5BI: Problem Set 11
Double and triple integrals

Suppose that D is a bounded region in the (x, y)-plane. We say that D is of type I if it can be described in the form

$$D = \{(x, y) \in R^2 : a \leq x \leq b, \phi(x) \leq y \leq \psi(x)\},$$

and of type II if it is of the form

$$D = \{(x, y) \in R^2 : c \leq y \leq d, \phi(y) \leq x \leq \psi(y)\}.$$

We say that D is *elementary* if it is one of these two types.

Suppose, in addition, that $f : D \to \mathbb{R}$ is a continuous function. If D is an elementary region of type I, we set

$$\int\int_D f(x, y)dxdy = \int_a^b \left[\int_{\phi(x)}^{\psi(x)} f(x, y)dy\right] dx, \tag{1}$$

while if D is of type II, we set

$$\int\int_D f(x, y)dxdy = \int_c^d \left[\int_{\phi(y)}^{\psi(y)} f(x, y)dx\right] dy. \tag{2}$$

It can be shown that if a region D is of both type I and type II, the two expressions for the double integral (**??**) and (**??**) agree. If D can be divided up into a finite union of elementary regions $D_1, \ldots, D_k$ such that each intersection $D_i \cap D_j$ consists of finitely many curves, then

$$\int\int_D f(x, y)dxdy = \int\int_{D_1} f(x, y)dxdy + \ldots \int\int_{D_k} f(x, y)dxdy.$$

It can be shown that the result obtained is independent of the way in which D is divided up into a finite disjoint union of elementary regions.

The double integral has many possible interpretations. Some of the most important are these:

First, if $f(x, y) \equiv 1$, then

$$\int\int_D 1dxdy = (\text{area of } D).$$

If $f(x, y) \geq 0$, then

$$\int\int_D f(x, y) dx dy$$

is the volume of the region

$$E = \{(x, y, z) \in R^3 : (x, y) \in D, 0 \leq z \leq f(x, y)\}.$$

If $f(x, y)$ represents mass density at (x, y), then the double integral

$$\int\int_D f(x, y) dx dy$$

is the total mass of D. Finally, we can use the double integral to compute the average value of f on D, by means of the formula,

$$(\text{Average value of } f \text{ on } D) = \frac{\int\int_D f(x, y) dx dy}{\int\int_D 1 dx dy}.$$

Problem 11.1. Let $D = \{(x, y) \in R^2 : x^2 \leq y \leq 2 - x^2\}$.

a. Find the area of D.

b. Find the center of mass of D.

c. Find the average value of the function $f(x, y) = x^2 + y^2$ on D.

d. Find the volume of the part of the paraboloid $z = x^2 + y^2$ which lies over D.

If E is a bounded region in (x, y, z)-space say that E is of type I if it can be described in the form

$$E = \{(x, y, z) \in R^3 : (x, y) \in D, \phi(x, y) \leq z \leq \psi(x, y)\},$$

for some elementary region D in the (x, y)-plane. In this case

$$\int\int\int_E f(x, y, z) dx dy dz = \int\int_D \left[\int_{\phi(x,y)}^{\psi(x,y)} f(x, y, z) dz\right] dx dy. \tag{3}$$

Similarly, E is of type II if it can be described in the form

$$E = \{(x, y, z) \in R^3 : (x, z) \in D, \phi(x, z) \leq y \leq \psi(x, z)\},$$

for some elementary region D in the (x, z)-plane, in which case we set

$$\int\int\int_E f(x, y, z) dx dy dz = \int\int_D \left[\int_{\phi(x,z)}^{\psi(x,z)} f(x, y, z) dy\right] dx dz, \tag{4}$$

and E is of type III if it can be described in the form

$$E = \{(x, y, z) \in R^3 : (y, z) \in D, \phi(y, z) \leq x \leq \psi(y, z)\},$$

for some elementary region D in the (y, z)-plane, in which case we set

$$\int\int\int_E f(x,y,z)dxdydz = \int\int_D \left[\int_{\phi(y,z)}^{\psi(y,z)} f(x,y,z)dx\right] dydz. \tag{5}$$

If the three-dimensional region E can be divided up into a finite union of elementary regions $E_1, \ldots, E_k$ such that each intersection $E_i \cap E_j$ consists of finitely many surfaces, then we can define the integral of f over E by the formula

$$\int\int\int_E f(x,y,z)dxdydz$$
$$= \int\int\int_{E_1} f(x,y,z)dxdydz + \ldots + \int\int\int_{E_k} f(x,y,z)dxdydz.$$

Like the double integral, the triple integral has many possible interpretations, depending on the context: volume, mass, average value, ...

Problem 11.2. Let $E = \{(x,y,z) \in R^3 : x + y + z \leq 1, x \geq 0, y \geq 0, z \geq 0\}$.

a. Find the volume of E.

b. Find the center of mass of E.

c. Find the average value of the function $f(x,y,z) = x^2 + y^2 + z^2$ on E.

A *differential* in the plane is an expression of the form

$$M(x,y)dx + N(x,y)dy,$$

where $M(x,y)$ and $N(x,y)$ are smooth functions. As we saw in Problem Set 10, differentials can be integrated along directed smooth curvers. Green's Theorem relates double integrals to line integrals:

Green's Theorem. *Let D be a bounded region in the (x,y)-plane, bounded by a piecewise smooth curve ∂D, directed so that as it is traversed in the positive direction, the region D lies on the left. Let $M(x,y)dx + N(x,y)dy$ be a differential on $D \cup \partial D$ whose component functions M and N are smooth. Then*

$$\int_{\partial D} Mdx + Ndy = \int\int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

Problem 11.3. Use Green's Theorem to evaluate the line integral

$$\int_{\mathbf{C}} (e^{-x^2} dx + xdy),$$

where $\mathbf{C}$ is the unit circle $x^2 + y^2 = 1$ traversed once in the counterclockwise direction. Hint: First reduce the line integral to a double integral and then evaluate the double integral.

Conversely, Green's theorem is often useful in evaluating double integrals. For example, suppose we want a formula for the area of a region D bounded by a smooth closed curve $\mathbf{C}$. We need only find functions $M(x,y)$ and $N(x,y)$ so that

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

Then

$$\text{Area of } D = \int\int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \int_{\mathbf{C}} Mdx + Ndy.$$

For example, we can set $M = -y$ and $N = 0$, to obtain the formula

$$\text{Area of } D = \int_{\mathbf{C}} -ydx,$$

or $M = -(1/2)y$ and $N = (1/2)x$ to obtain the formula

$$\text{Area of } D = \int_{\mathbf{C}} [-(1/2)ydx + (1/2)xdy]. \tag{6}$$

Problem 11.4. Use Green's Theorem to determine the area of the region in the (x,y)-plane bounded by the curve $x^{(2/3)} + y^{(2/3)} = 1$. Hint: We can parametrize this curve by

$$\mathbf{x}(t) = \begin{pmatrix} \cos^3 t \\ \sin^3 t \end{pmatrix}, \quad t \in [0, 2\pi],$$

and use formula (**??**). Can you sketch the curve?