Math 5BI: Problem Set 10
Line integrals of vector fields

Suppose that $\mathbf{C}$ is a *directed* regular curve in the plane, that is, a curve with a sense of direction. Let $\mathbf{x} : [a, b] \to R^2$ is a parametrization such that as $\mathbf{x}/dt$ is never zero and as t increases, $\mathbf{C}$ is traversed in the positive direction. The orientation picks our a unit tangent vector to $\mathbf{C}$, the unit-length vector

$$\mathbf{T}(t) = \frac{\mathbf{x}'(t)}{|\mathbf{x}'(t)|}.$$

Given a smooth vector field

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j},$$

we can form the *line integral* of the vector field $\mathbf{F}$ along $\mathbf{C}$,

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T} ds.$$

There are some very useful alternate notations for this line integral. Since

$$\mathbf{T} ds = \mathbf{T}\frac{ds}{dt}dt = \mathbf{x}'(t)dt = d\mathbf{x} = dx\mathbf{i} + dy\mathbf{j},$$

we can write

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T} ds = \int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{\mathbf{C}} M dx + N dy.$$

An important interpretation of this line integral occurs in physics. If $\mathbf{F}$ represents the force acting on a body which moves along the parametrized directed curve $\mathbf{x} : [a, b] \to \mathbb{R}^n$, then the line integral

$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{x}$$

represent the total *work* performed by the force on the body.

Problem 10.1. a. Suppose that $\mathbf{F}(x, y) = xy\mathbf{i} + (y - 3)\mathbf{j}$ and that $\mathbf{C}$ is the part of the parabola parametrized by

$$\mathbf{x} : [-1, 1] \to \mathbb{R}^2, \quad \mathbf{x}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

and directed from left to right. Show that

$$\mathbf{F} \cdot \mathbf{T} ds = xydx + (y-3)dy.$$

b. Find

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T} ds.$$

It is particularly easy to calculate the line integral of a gradient along a directed curve. Indeed, the "fundamental theorem of calculus," which asserts that differentiation and integration are inverse processes, can be generalized to the context of line integrals:

Theorem. *Let* $\mathbf{x} : [a,b] \to \mathbb{R}^2$ *be a parametrization of a directed curve* $\mathbf{C}$ *from the point* (x_0, y_0) *to the point* (x_1, y_1)*. If* $f(x,y)$ *is any smooth function, then*

$$\int_{\mathbf{C}} \nabla f \cdot d\mathbf{x} = f(x_1, y_1) - f(x_0, y_0).$$

Problem 10.2. To prove this theorem, one uses the chain rule and the usual version of the fundamental theorem of calculus. Prove the theorem by showing that

$$\int_{\mathbf{C}} \nabla f \cdot d\mathbf{x} = \cdots = f(x_1, y_1) - f(x_0, y_0).$$

The above ideas can be extended quite easily to directed curves in $\mathbb{R}^n$. If $\mathbf{x} : [a,b] \to \mathbb{R}^n$ is a parametrization of a regular curve $\mathbf{C}$ in $\mathbb{R}^n$ and

$$\mathbf{F}(x_1, \ldots, x_n) = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \cdots \\ f_n(x_1, \ldots, x_n) \end{pmatrix},$$

then the line integral

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T} ds = \int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{\mathbf{C}} f_1 dx_1 + \ldots + f_n dx_n$$

can be calculated by simply expressing the last integral on the right in terms of the parameter t,

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \left[f_1(x_1(t), \ldots, x_n(t)) \frac{dx_1}{dt} + \cdots + f_n(x_1(t), \ldots, x_n(t)) \frac{dx_n}{dt} \right] dt.$$

The above theorem can also be generalized to the case where $\mathbb{R}^2$ is replaced by $\mathbb{R}^n$. Thus if $\mathbf{x} : [a,b] \to \mathbb{R}^n$ is a parametrization of $\mathbf{C}$,

$$\int_{\mathbf{C}} (\nabla f) \cdot \mathbf{T} ds = f(\mathbf{x}(b)) - f(\mathbf{x}(a)).$$

Problem 10.3. a. Evaluate the line integral

$$\int_{\mathbf{C}} y dx - x dy,$$

where $\mathbf{C}$ is the straight line segment in $\mathbb{R}^2$ from $(0,0)$ to $(3,4)$.

b. Evaluate the same line integral in the case where $\mathbf{C}$ is the circle of radius one in $\mathbb{R}^2$ centered at the origin, directed counterclockwise.

c. Evaluate the same line integral in the case where $\mathbf{C}$ is the curve parametrized by

$$\mathbf{x} : [0,1] \to \mathbb{R}^2, \quad \text{where} \quad \mathbf{x}(t) = (t, t^2).$$

Problem 10.4. a. Evaluate the line integral

$$\int_{\mathbf{C}} z dx - x dy + y dz,$$

in the case where $\mathbf{C}$ is the curve in $\mathbb{R}^3$ parametrized by

$$\mathbf{x} : [0,1] \to \mathbb{R}^3, \quad \text{where} \quad \mathbf{x}(t) = (t, t^2, t^3).$$

b. Evaluate the line integral

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T} ds,$$

where $F(x,y,z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$ and $\mathbf{C}$ is the straight line segment in $\mathbb{R}^3$ from $(0,0,0)$ to $(1,2,2)$.

Problem 10.5. a. Evaluate the line integral

$$\int_{\mathbf{C}} \nabla f \cdot d\mathbf{x},$$

where $f(x,y) = x + 3y$ and $\mathbf{C}$ is the directed straight line segment in $\mathbb{R}^2$ from $(0,0)$ to $(3,4)$.

b. Evaluate the line integral

$$\int_{\mathbf{C}} \nabla f \cdot d\mathbf{x},$$

where $f(x,y) = x + 3y$ and $\mathbf{C}$ is the circle of radius one in $\mathbb{R}^2$ centered at the origin, directed counterclockwise.

c. Evaluate the line integral

$$\int_{\mathbf{C}} \nabla f \cdot d\mathbf{x},$$

where $f(x,y,z) = x^2 + y^2 + z^2$ and $\mathbf{C}$ is the directed curve in $\mathbb{R}^3$ parametrized by

$$\mathbf{x} : [0,1] \to \mathbb{R}^3, \quad \text{where} \quad \mathbf{x}(t) = (t, t^2, t^3).$$