Math 5BI: Problem Set I
Derivatives of functions of several variables

Recall that in first year calculus, one of the things the the derivative determines is the equation of the line tangent to the graph of a function. If $f(x)$ is a differentiable function and x_0 is a value for the variable x, then the line tangent to the graph of the curve $y = f(x)$ is given by the equation

$$y = f(x_0) + \left(\frac{df}{dx}\right)(x_0)(x - x_0). \tag{1}$$

In this equation x_0, $f(x_0)$, and $\left(\frac{df}{dx}\right)(x_0)$ are constants, while x and y are variables. The function

$$L(x) = f(x_0) + \left(\frac{df}{dx}\right)(x_0)(x - x_0)$$

will be called the *linearization* of $f(x)$ at x_0. (Strictly speaking, it is not usually a linear function, but an *affine function*, which is the same thing as a linear function plus a constant.) The *tangent line* to the graph of $y = f(x)$ at $(x_0, f(x_0))$ is the graph of the linearization $y = L(x)$ of $f(x)$ at x_0.

For example, if $f(x) = x^3 + 1$, then $(1, 2)$ lies on the graph of $y = f(x)$, and the local linearization at this point is $L(x) = 2 + 3(x - 1)$. This function $L(x)$ is the best linear approximation to $y = f(x)$ at the point $(1, 2)$.

Problem 1.1. Suppose $z = f(x, y)$ is the function of two variables given by $f(x, y) = x^2 + y$. Then $(2, 5, 9) \in \mathbb{R}^3$ lies on the graph of $z = f(x, y)$. Extend your thinking from the one-variable case to the two-variable case and figure out what the local linearization of $z = f(x, y)$ should be at the point $(2, 5, 9)$.

To visualize the meaning of the linearization of a function $f(x, y)$ at a given point (x_0, y_0), we consider two parametrized curves which lie on the graph of f and intersect at the point $(x_0, y_0, f(x_0, y_0))$:

$$\gamma_1 : \mathbb{R} \to \mathbb{R}^3, \qquad \gamma_1(x) = (x, y_0, f(x, y_0)),$$

$$\gamma_2 : \mathbb{R} \to \mathbb{R}^3, \qquad \gamma_2(y) = (x_0, y, f(x_0, y)).$$

The parameters on these two curves are x and y, respectively. The velocity vectors to these curves are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ h_1'(x_0) \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ h_2'(y_0) \end{pmatrix}. \tag{2}$$

where $h_1(x) = f(x, y_0)$ and $h_2(y) = f(x_0, y)$

Problem 1.2. (a) Explain, in geometric terms, what the above discussion means.
(b) In the above notation we denote $\gamma_1'(x_0)$ by $(\partial f/\partial x)(x_0, y_0)$ and $\gamma_2'(y_0)$ by $(\partial f/\partial y)(x_0, y_0)$. If you used limits, how would you define these?

If $z = f(x, y)$, where f is a smooth function and x_0 and y_0 are values for the variables x and y, then the linearization of f at (x_0, y_0) is the affine function (linear function plus a constant) whose graph is the tangent plane to the graph of f at the point $(x_0, y_0, f(x_0, y_0))$.

To find this linearization, we need to take *partial derivatives*. The partial derivative of $f(x, y)$ with respect to x at (x_0, y_0) is given by the formula

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

Thus the partial derivative of $f(x, y)$ with respect to x at (x_0, y_0) is the ordinary derivative of the function

$$x \mapsto f(x, y_0) \quad \text{at the point} \quad x_0.$$

Similarly, the partial derivative of $f(x, y)$ with respect to y at (x_0, y_0) is

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}.$$

Definition. Let $f(x, y)$ be a function of two variables. We say that f is *continuously differentiable* if it possesses partial derivatives $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ which vary continuously with the point (x, y).

Problem 1.3. Consider the function $z = f(x, y) = x^2 + y^2$.
a. For this choice of f, find the velocity vector to the curve

$$\gamma_1 : \mathbb{R} \to \mathbb{R}^3, \qquad \gamma_1(x) = (x, 2, f(x, 2))$$

at the point $x = x_0 = 3$.

b. Find the velocity vector to the curve

$$\gamma_2 : \mathbb{R} \to \mathbb{R}^3, \qquad \gamma_2(y) = (3, y, f(3, y))$$

at the point $y = y_0 = 2$.

c. Find a vector perpendicular to the surface $z = x^2 + y^2$ at the point $(3, 2, 13)$.

d. Find the equation of the plane tangent to the surface $z = x^2 + y^2$ at the point $(3, 2, 13)$ in two ways. First use the perpendicular vector and the dot product. Then use the ideas involving the local linearization developed above. Explain why they coincide.

Problem 1.4. Suppose that $f(x, y)$ is a continuously differentiable function of two variables. Based upon your work on the above problems,write up careful notes explaining how to find the plane tangent to the surface $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ and the local linearization of f at the point(x_0, y_0).

It follows from the previous problem that the tangent plane to the surface $z = f(x, y)$ is horizontal if and only if

$$\left(\frac{\partial f}{\partial x}\right)(x_0, y_0) = \left(\frac{\partial f}{\partial y}\right)(x_0, y_0) = 0. \tag{3}$$

A point satisfying (3) is called a *critical point* of the function f. If one thinks of the graph of f as representing a mountain range, then mountain peaks, mountain passes, and bottoms of lakes are critical points of f. In particular, if a function $f(x, y)$ assumes a local maximum or a local minimum at a point (x_0, y_0), then (x_0, y_0) must be a critical point for f.

This last fact can be used to find the candidates for maxima and minima of a given function of two variables.

Problem 1.5. a. Suppose two lines are given in parametric form by

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \qquad \mathbf{y}(u) = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} + u \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

To find the distance between these two lines, we first let

$$f(t, u) = (\text{distance from the point } \mathbf{x}(t) \text{ to the point } \mathbf{y}(u))^2.$$

Find the critical points for the function $f(t, u)$.

Problem 1.6. Let Π be the plane in R^3 defined in parametric form by

$$\mathbf{x}(s, t) = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + s \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

a. Find the critical points of the function

$$f(s, t) = (\text{distance from the point } (3, 5, 1) \text{ to the point } \mathbf{x}(s, t))^2.$$

b. Find the distance from the plane Π to the point $(3, 5, 1)$.