Math 5BI Introductory Problem Set 0

Problem 0.1. The diagram below gives lines of constant temperature along one side of a heated room through an entire winter day. Time is on the vertical axis and distance along the wall is on the horizontal axis.

Along the wall of the room there is a heater, a thermostat, and a window. Using the diagram, find out as much information about the room that day as you can. For example, find the locations of the heater, thermostat, window, and tell the story of what happened that day. For example, what temperature you think the thermostat was set at, and when was the heat turned on. Was the window ever opened?, and so forth. Be able to justify you thinking.

In Math 3C and 5A you studied vectors, but mostly from an algebraic point of view (for example, you studied linear independence, span, dimension of vector spaces and so forth.) This type of information is important for 5B as well, but we also need some geometric properties. The next two problems introduce the dot product, and the problems that follow introduce the cross product in $\mathbf{R}^3$.

Problem 0.2. If $\mathbf{v}, \mathbf{w}$ are vectors in $\mathbf{R}^2$ or $\mathbf{R}^3$, then their dot product is defined by

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \, \|\mathbf{w}\| \cos\theta$$

where $\|\mathbf{v}\|$ is the length of $\mathbf{v}$ and where θ is the angle between the two vectors.

Now, nobody wants to compute the dot product using the length formula and the cosine. So these next questions will lead you to a different way.
(a) Using basic trigonometry, give a geometric interpretation of $\|\mathbf{w}\|\cos\theta$ as the length a vector in the direction of $\mathbf{v}$.
(b) Now extend this understanding to $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \, \|\mathbf{w}\| \cos\theta$. Draw a picture.
(c) What does $\mathbf{v} \cdot \mathbf{w} = 0$ mean?

Problem 0.3. (a) Using Probem 0.2 verify the following three rules

$\mathbf{v} \cdot (\lambda \mathbf{w}) = \lambda(\mathbf{v} \cdot \mathbf{w})$ for $\lambda \in \mathbf{R}$.
$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
$(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = (\mathbf{v} \cdot \mathbf{u}) + (\mathbf{w} \cdot \mathbf{u})$

(b) Show that in $\mathbf{R}^2$ that $(a, b) \cdot (c, d) = ac + bd$. Does this generalize? (Hint: use the rules in (a), don't use the cosine!)

Problem 0.4. a. Under what conditions is the vector

$$\mathbf{x} - (3, 7, 1) = (x, y, z) - (3, 7, 1)$$

perpendicular to the vector $(5, 2, 2)$?

b. Find an equation that represents the plane which contains the point $\mathbf{p} = (3, 7, 1)$ and is perpendicular to the vector $(5, 2, 2)$.

The standard basis for $\mathbf{R}^3$ is $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ where

$$\mathbf{i} = (1, 0, 0), \qquad \mathbf{j} = (0, 1, 0), \qquad \mathbf{k} = (0, 0, 1).$$

Can you show that it actually is a basis for the vector space $\mathbf{R}^3$?

Definition. Suppose that $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are two vectors in $\mathbf{R}^3$. The *cross product* of $\mathbf{v}$ and $\mathbf{w}$ is the vector

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} + \begin{vmatrix} v_3 & v_1 \\ w_3 & w_1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k}.$$

Problem 0.5. a. Find the cross product $\mathbf{v} \times \mathbf{w}$ of the vectors $\mathbf{v} = (1, 3, 6)$ and $\mathbf{w} = (2, 1, 5)$.

b. Show that the result $\mathbf{v} \times \mathbf{w}$ is perpendicular to $\mathbf{v} = (1, 3, 6)$ and $\mathbf{w} = (2, 1, 5)$.

c. Show that if $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are two linearly independent vectors in $\mathbf{R}^3$, then $\mathbf{v} \times \mathbf{w}$ is perpendicular to $\mathbf{v}$ and $\mathbf{w}$.

Problem 0.6. a. Find a vector that is perpendicular to the vectors $(1, 0, 6)$ and $(0, 1, 4)$.

b. Find an equation of the plane which is tangent to the vectors $(1, 0, 6)$ and $(0, 1, 4)$ and contains the point $(3, 2, 13)$.

Problem 0.7. a. Show that if $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are two vectors in $\mathbf{R}^3$, then

$$|\mathbf{v} \times \mathbf{w}|^2 + |\mathbf{v} \cdot \mathbf{w}|^2 = |\mathbf{v}|^2 |\mathbf{w}|^2.$$

b. Find a formula for he vectors $|\mathbf{v} \times \mathbf{w}|$ in terms of $|\mathbf{v}|$, $|\mathbf{w}|$ and the angle θ between $\mathbf{v}$ and $\mathbf{w}$.