Name: **Perm number:**

Final exam – take-home part

Due date: Thursday, June 11th

Surface areas

Sometimes surfaces in $\mathbb{R}^3$ are conveniently represented as graphs of functions, sometimes as the images of smooth maps, called parametrizations.

For example, the *paraboloid of revolution* in $\mathbb{R}^3$, defined as the set of points which satisfy the equation

$$z = x^2 + y^2,$$

can be thought of as the graph of the function $f(x, y) = x^2 + y^2$. But we can also regard it as the image of the map

$$\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3 \quad \text{by} \quad \mathbf{x}(u, v) = \begin{pmatrix} u \\ v \\ u^2 + v^2 \end{pmatrix}.$$

The map $\mathbf{x}$ is called a parametrization for the paraboloid of revolution.

A *parametrization* of a smooth surface S is simply a smooth one-to-one map $\mathbf{x}$ from a domain D in the (u, v) plane onto S.

A plane passing through the origin of $\mathbb{R}^3$ is simply a two-dimensional linear subspace—it can be parametrized by the mapping $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ given by the formula

$$\mathbf{x}(u, v) = u\mathbf{b}_1 + v\mathbf{b}_2,$$

where $\mathbf{b}_1, \mathbf{b}_2$ is a basis for the plane. The plane that passes through the point $\mathbf{p}$ and is parallel to the linearly independent vectors $\mathbf{b}_1, \mathbf{b}_2$ is parametrized by $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$, where

$$\mathbf{x}(u, v) = \mathbf{p} + u\mathbf{b}_1 + v\mathbf{b}_2.$$

(1) **(10 points)** a. Show that $\{(1, 0, 1), (0, 1, 1)\}$ is a basis for the linear subspace of $\mathbb{R}^3$ defined as the set of solutions to the homogeneous linear equation $x + y - z = 0$.

b. Find a parametrization for the plane $x + y - z = 7$.

The sphere of radius a centered at the origin, $x^2 + y^2 + z^2 = a^2$, can be parametrized in terms of spherical coordinates:

$$\begin{cases} x = r\cos\theta\sin\phi, \\ y = r\sin\theta\sin\phi, \\ z = r\cos\phi. \end{cases}$$

In these spherical coordinates, the sphere is represented by the equation $r = a$, so we can parametrize the sphere by $\mathbf{x} : D \to \mathbb{R}^3$, where

$$D = \{(\theta, \phi) \in \mathbb{R}^2 : 0 < \theta < 2\pi, 0 < \phi < \pi\}$$

and

$$\mathbf{x}(\theta, \phi) = \begin{pmatrix} a\cos\theta\sin\phi \\ a\sin\theta\sin\phi \\ a\cos\phi \end{pmatrix}.$$

Actually, this parametrization does not cover the entire sphere. It misses the "prime meridian," a subset of zero area, which will not affect our subsequent calculations of surface integrals.

(2) **(10 points)** Find a parametrization of the ellipsoid,

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1,$$

where a, b, and c are positive, by introducing the new variables

$$u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c}.$$

so that the equation of the ellipsoid simplifies to $u^2 + v^2 + w^2 = 1$, and then use our previous parametrization of the sphere.

A surface obtained from a smooth curve in the right half of the (x, z)-plane by rotating the curve about the z-axis is called a *surface of revolution*. Surfaces of revolution can be conveniently parametrized by means of cylindrical coordinates. For example, suppose that the curve in the (x, z)-plane is the catenary $x = \cosh z$. In terms of cylindrical coordinates

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ z = z. \end{cases}$$

the surface revolution generated by the catenary is represented by the equation $r = \cosh z$. This surface is called the *catenoid*. We can parametrize the catenoid by setting $z = u$ and $\theta = v$.

(3) **(10 points)** a. Sketch the catenoid.

b. Find a parametrization $\mathbf{x} : D \to \mathbb{R}^3$ of the catenoid, where

$$D = \{(u, v) \in \mathbb{R}^2 : 0 \leq v < 2\pi\}.$$

Now we turn to the problem of calculating surface area. We start with the observation that the area of the parallelogram spanned by two vectors $\mathbf{v}$ and $\mathbf{w}$ is simply the length of their cross product,

$$\text{Area} = |\mathbf{v} \times \mathbf{w}|.$$

Suppose now that $\mathbf{x} : D \to \mathbb{R}^3$ is the parametrization of a surface S, and that $(u_0, v_0) \in D$. Let $\square$ denote the rectangular region in D with corners at (u_0, v_0), $(u_0 + du, v_0)$, $(u_0, v_0 + dv)$, and $(u_0 + du, v_0 + dv)$. The linearization of $\mathbf{x}$ at (u_0, v_0) is the affine mapping

$$\mathbf{L}(\mathbf{x}) = \mathbf{x}(u_0, v_0) + \frac{\partial \mathbf{x}}{\partial u}(u - u_0) + \frac{\partial \mathbf{x}}{\partial v}(v - v_0).$$

Under this affine mapping

$$(u_0, v_0) \mapsto \mathbf{x}(u_0, v_0), \quad (u_0 + du, v_0) \mapsto \mathbf{x}(u_0, v_0) + \frac{\partial \mathbf{x}}{\partial u} du,$$

$$(u_0, v_0 + dv) \mapsto \mathbf{x}(u_0, v_0) + \frac{\partial \mathbf{x}}{\partial v} dv, \quad (u_0 + du, v_0 + dv) \mapsto \mathbf{x}(u_0, v_0) + \frac{\partial \mathbf{x}}{\partial u} du + \frac{\partial \mathbf{x}}{\partial v} dv.$$

The four image points are the corners of the parallelogram located at $\mathbf{x}(u_0, v_0)$ and spanned by

$$\frac{\partial \mathbf{x}}{\partial u}(u_0, v_0) du \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial v}(u_0, v_0) dv,$$

a parallelogram which has area

$$dA = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| du dv.$$

Since the linearization closely approximates the parametrization $\mathbf{x} : D \to R^3$ near (u_0, v_0), the area of $\mathbf{x}(\square)$ is closely approximated by

$$dA = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| du dv.$$

If we divide D up into many small rectangles like $\square$ and add up their contributions to the area, we obtain the following formula for the surface area of a surface S parametrized by $\mathbf{x} : D \to \mathbb{R}^3$:

$$\text{Surface area of } S = \int \int_D \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| du dv.$$

For example suppose that we want to find the area of the sphere which is defined by the equation $x^2 + y^2 + z^2 = a^2$. We can use the parametrization $\mathbf{x} : D \to \mathbb{R}^3$, where

$$D = \{(\theta, \phi) \in R^2 : 0 < \theta < 2\pi, 0 < \phi < \pi\}$$

and

$$\mathbf{x}(\theta, \phi) = \begin{pmatrix} a \cos\theta \sin\phi \\ a \sin\theta \sin\phi \\ a \cos\phi \end{pmatrix}.$$

(4) **(5 points)** Find the surface area of the sphere of radius a.

(5) **(5 points)** Find the surface area of that part of the paraboloid $z = x^2 + y^2$ which lies inside the cylinder $x^2 + y^2 = 1$.

(6) **(5 points)** Find the surface area of that part of the cone $x^2 + y^2 = z^2$ which lies between the planes $z = 0$ and $z = 2$.

(7) **(5 points)** Find the surface area of that part of the catenoid $x^2 + y^2 = \sinh^2 z$ which lies between the planes $z = -1$ and $z = 1$.

(8) **(5 points)** Let S be the *torus* defined by the equation

$$(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1,$$

with the parametrization $\mathbf{x} : D \to S$, defined by

$$\mathbf{x}(u, v) = \begin{pmatrix} (2 + \cos v) \cos u \\ (2 + \cos v) \sin u \\ \sin v \end{pmatrix},$$

where

$$D = \{(u, v) \in \mathbb{R}^2 : -\pi < u < \pi, -\pi < v < \pi\}.$$

Find the surface area of S.

If $\mathbf{x} : D \to R^3$ is the parametrization of a surface $\mathbf{S}$ and $f(x, y, z)$ is any continuous function of three variables, the *surface integral* of f over $\mathbf{S}$ is given by the formula

$$\int\int_{\mathbf{S}} f(x, y, z) dA = \int\int_D f(x(u, v), y(u, v), z(u, v)) \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| dudv.$$

In more advanced texts it is shown that the integral thus defined is independent of parametrization.

(9) **(5 points)** If

$$\mathbf{S} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = a^2, z \geq 0\}.$$

evaluate the surface integral

$$\int\int_{\mathbf{S}} z dA.$$

Flux integrals

Suppose now that we have a continuous choice of unit-normal $\mathbf{N}$ to the surface $\mathbf{S}$. Such a continuous choice of unit-normal is called an *orientation* of $\mathbf{S}$. If

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

is a smooth vector field on R^3 the *flux* of $\mathbf{F}$ through $\mathbf{S}$ is given by the surface integral

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA.$$

Calculation of flux integrals is simpler than might be expected, because

$$\mathbf{N} dA = \frac{\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}}{\left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right|} \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| dudv = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} dudv,$$

and hence

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA = \int\int_D \mathbf{F}(x(u, v), y(u, v), z(u, v)) \cdot \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right) dudv.$$

(10) **(bonus 20 points)** If $\mathbf{S}$ is the hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$, and

$$\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + z^3\mathbf{k},$$

evaluate

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA.$$

A physical picture for the flux integral: Suppose that a fluid is flowing throughout (x, y, z)-space with velocity $\mathbf{V}(x, y, z)$ and density $\rho(x, y, z)$. In this case, fluid flow is represented by the vector field

$$\mathbf{F} = \rho \mathbf{V},$$

and the surface integral

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA \tag{1}$$

represents the rate at which the fluid is flowing accross $\mathbf{S}$ in the direction of $\mathbf{N}$.

To see this, note first that the rate at which fluid flows across a small piece of $\mathbf{S}$ of surface area dA is

$$(\text{density})(\text{normal component of velocity}) dA = \rho \mathbf{V} \cdot \mathbf{N} dA.$$

If we add up the contributions of all the small area elements, we obtain the integral (1).

Gradient operator

The gradient operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

operates not only on functions, but on vector fields in two different ways. If

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

is a smooth vector field, its *divergence* is the function

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \cdot (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

while its *curl* is the vector field

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ Q & R \end{vmatrix} \mathbf{i} + \begin{vmatrix} \partial/\partial z & \partial/\partial x \\ R & P \end{vmatrix} \mathbf{j} + \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ P & Q \end{vmatrix} \mathbf{k}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

The geometrical and physical interpretations of the divergence and the curl come from the divergence theorem and Stokes's theorem.

The Divergence Theorem. *Let D be a region in (x, y, z)-space which is bounded by a piecewise smooth surface* $\mathbf{S}$. *Let* $\mathbf{N}$ *be the outward-pointing unit normal to* $\mathbf{S}$. *If* $\mathbf{F}(x, y, z)$ *is a vector field which is smooth on D and its boundary, then*

$$\int\int\int_D (\nabla \cdot \mathbf{F}) dxdydz = \int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA.$$

The proof of the divergence theorem is very similar to the proof of Green's theorem. Like Green's theorem, the divergence theorem can be used to reduce a complicated surface integral to a simpler volume integral, or a complicated volume integral to a simpler surface integral.

(11) **(10 points)** a. Find the divergence of the vector field

$$\mathbf{F}(x, y, z) = \log(y^2 + z^2 + 1)\mathbf{i} + y\mathbf{j} + (\sin x \cos y)\mathbf{k}.$$

b. Use the divergence theorem to evaluate the flux integral

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA,$$

where $\mathbf{S}$ is the boundary of the cube

$$-1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1,$$

and $\mathbf{N}$ is the outward-pointing unit normal.

(12) **(10 points)** a. Find the divergence of the vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

b. Let $\mathbf{S}$ be the unit sphere defined by the equation $x^2 + y^2 + z^2$. Show that the evaluation of bfF at a point of the unit sphere is the outward=pointing unit normal to the sphere. Show that

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA = \text{area of } \mathbf{S}.$$

c. Use the divergence theorem to evaluate

$$\int\int\int_D (\nabla \cdot \mathbf{F}) dxdydz,$$

where D is the unit ball bounded by the unit sphere $x^2 + y^2 + z^2 = 1$. Use your result to calculate the volume of the unit ball.

Remark. The reason the divergence theorem is so important is that it can be used to derive many of the important partial differential equations of mathematical physics. For example, we can use it to derive the *equation of continuity* from fluid mechanics.

Indeed, suppose that a fluid is flowing throughout (x, y, z)-space with velocity $\mathbf{V}(x, y, z, t)$ and density $\rho(x, y, z, t)$. If we represent the fluid flow by the vector field

$$\mathbf{F} = \rho\mathbf{V},$$

then the surface integral

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA$$

represents the rate at which the fluid is flowing accross $\mathbf{S}$ in the direction of $\mathbf{N}$.

We assume that no fluid is being created or destroyed. Then the rate of change of the mass of fluid within D is given by two expressions,

$$\int\int\int_D \frac{\partial \rho}{\partial t}(x, y, z, t) dxdydz$$

and

$$-\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA.$$

It follows from the divergence theorem that the second of these expressions equals

$$-\int\int\int_D \nabla \cdot \mathbf{F}(x,y,z,t)dxdydz.$$

Thus

$$\int\int\int_D \frac{\partial \rho}{\partial t} dxdydz = -\int\int\int_D \nabla \cdot \mathbf{F} dxdydz.$$

Since this equation must hold for *every* region D in (x,y,z)-space, we conclude that the integrands must be equal,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{F} = -\nabla \cdot (\rho \mathbf{V}).$$

Thus we obtain the equation of continuity,

(2) $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0.$$

The other major integral theorem from vector calculus is:

Stokes's Theorem. *If* $\mathbf{S}$ *is an oriented smooth surface in* R^3 *bounded by a piecewise smooth curve* $\partial \mathbf{S}$*, and* $\mathbf{F}$ *is a smooth vector field on* R^3*, then*

$$\int\int_{\mathbf{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{N} dA = \int_{\partial \mathbf{S}} \mathbf{F} \cdot \mathbf{T} ds,$$

where $\mathbf{N}$ *is the unit normal chosen by the orientation and* $\mathbf{T}$ *is the unit tangent to* $\partial \mathbf{S}$ *chosen so that* $\mathbf{N} \times \mathbf{T}$ *points into* $\mathbf{S}$.

Once again, Stokes's theorem can be used in two directions, to reduce a complicated line integral to a simpler surface integral or a complicated surface integral to a simpler line integral.

(13) **(20 points)** a. Find a parametrization for the circle $\mathbf{C}$ of radius one lying in the plane $z = 4$ and centered on the line $x = y = 0$, oriented so that it is traversed counterclockwise when viewed from above.

b. Find the curl of the vector field

$$\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{j} + 12\mathbf{k}.$$

c. Use Stokes's theorem to evaluate the line integral

$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{x}.$$

Remark. Stokes's theorem gives rise to a geometric interpretation of curl. Indeed, let (x_0, y_0, z_0) be a given point in R^3, $\mathbf{N}$ a unit-length vector located at (x_0, y_0, z_0), Π the plane through (x_0, y_0, z_0) which is perpendicular to $\mathbf{N}$. Let D_ϵ be the disk in Π of radius ϵ centered at (x_0, y_0, z_0), ∂D_ϵ the circle in Π of radius ϵ centered at (x_0, y_0, z_0). Then

$$\begin{aligned}(\nabla \times \mathbf{F})(x_0, y_0, z_0) \cdot \mathbf{N} &= \lim_{\epsilon \to 0} \frac{\int\int_{D_\epsilon} (\nabla \times \mathbf{F}) \cdot \mathbf{N} dA}{\text{Area of } D_\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\int_{\partial D_\epsilon} \mathbf{F} \cdot \mathbf{T} ds}{\text{Area of } D_\epsilon} = \lim_{\epsilon \to 0} \frac{\text{Rate of circulation of } \mathbf{F} \text{ about } \partial D_\epsilon}{\text{Area of } D_\epsilon}.\end{aligned}$$

Thus $(\nabla \times \mathbf{F})(x_0, y_0, z_0)$ measures the rate of circulation of the fluid flow represented by $\mathbf{F}$ near (x_0, y_0, z_0).