Name: **Perm number:**

Final – take-home part

1. **(100 points)** Let $A \in \mathbb{R}^n_n$ be a matrix of a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, whose characteristic polynomial can be factored as follows:
$$(\lambda - a_1)^{k_1} \cdot (\lambda - a_2)^{k_2} \cdot \ldots \cdot (\lambda - a_m)^{k_m}$$
where $k_1 + k_2 + \ldots + k_m = n$, and suppose that $(v_{i1}, v_{i2}, \ldots, v_{il_i})$, for $1 \le i \le m$ is a basis of the eigenspace associated with the eigenvalue a_i, for $1 \le i \le m$, where $l_1 + l_2 + \ldots + l_m \le n$ In other words, we have m different eigenvalues $a_1, \ldots, a_m$, but dimensions of all eigenspaces might not add up to the dimension of the space $\mathbb{R}^n$.

For each eigenvalue a_i, $1 \le i \le m$, we proceed with the following algorithm, that will result with a sequence of l_i numbers $(r_{i1}, \ldots, r_{il_i})$, and a sequence of s_i vectors $(w_{i1}, \ldots, w_{is_i})$, where $s_i = r_{i1} + r_{i2} + \ldots + r_{il_i}$. In the description of the algorithm we use j to denote both the current step of the algorithm, and a number in the sequence $(w_{i1}, \ldots, w_{is_i})$ (for example $j = 7$ means we are at the 7th step, and are currently working with the vector w_{i7}), whereas t will be used to denote both the number of the last used vector from the basis $(v_{i1}, \ldots, v_{il_i})$, and a number in the sequence $(r_{i1}, \ldots, r_{il_i})$ (for example $t = 3$ means we last used vector v_{i3}, and are currently working with the number r_{i3}). We start with $j = 1$ and $t = 1$.

   (a) Denote $w_{ij} = v_{ij}$.
   (b) For the vector w_{ij} we consider the system of equations $T(w) = a_i w + w_{ij}$.
   (c) If the above system of equations has a solution w, we take $w_{i,j+1} = w$, increase j by 1, and go back to (2).
   (d) If the above system of equations does not have a solution, we take $r_{it} = j - (r_{i1} + \ldots + r_{i,t-1})$ (we assume that $r_{i0} = 0$ in the first step), increase j by 1, increase t by 1, and go back to (1).

   Apply the above algorithm to solve the following problems:

   (a) Test the algorithm for two linear operators, one whose matrix (in the standard basis) is
$$\begin{bmatrix} 6 & 2 & -2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix},$$
   and the second one whose matrix is:
$$\begin{bmatrix} 6 & 2 & 2 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$
   (b) Find matrices of the above operators in the bases (w_{11}, w_{21}, w_{22}) obtained in (a) for each operator.
   (c) Let, as before, $A \in \mathbb{R}^n_n$ be a matrix of a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, whose characteristic polynomial can be factored as follows:
$$(\lambda - a_1)^{k_1} \cdot (\lambda - a_2)^{k_2} \cdot \ldots \cdot (\lambda - a_m)^{k_m}$$
   where $k_1 + k_2 + \ldots + k_m = n$, and let, for $1 \le i \le m$, $(r_{i1}, \ldots, r_{il_i})$, and $(w_{i1}, \ldots, w_{is_i})$, where $s_i = r_{i1} + r_{i2} + \ldots + r_{il_i}$, be sequences of numbers and vectors obtained by using our algorithm for eigenvalues a_i. It can be, in fact, proven that $s_1 + s_2 + \ldots + s_m = n$, and that the vectors:
$$w_{11}, \ldots, w_{1s_1}, w_{21}, \ldots, w_{2s_2}, \ldots, w_{m1}, \ldots, w_{ms_m}$$
   form a basis of $\mathbb{R}^n$ – we will skip that proof, although you have already noticed that in the examples that you worked a while ago. A matrix
$$J_{it} = \begin{bmatrix} a & 1 & 0 & \cdots & 0 & 0 \\ 0 & a_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & a_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & a_i \end{bmatrix} \in \mathbb{R}^{r_{it}}_{r_{it}}$$

is called the **Jordan matrix** of degree r_{it} for the eigenvalue a_i.
Show that the matrix of T in the basis

$$w_{11}, \ldots, w_{1s_1}, w_{21}, \ldots, w_{2s_2}, \ldots, w_{m1}, \ldots, w_{ms_m}$$

is the following one:

$$\left[\begin{array}{c|c|c|c}
\begin{array}{c|c|c|c} J_{11} & 0 & \cdots & 0 \\ \hline 0 & J_{12} & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & J_{1l_1} \end{array} & 0 & \cdots & 0 \\ \hline
0 & \begin{array}{c|c|c|c} J_{21} & 0 & \cdots & 0 \\ \hline 0 & J_{22} & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & J_{2l_2} \end{array} & \cdots & 0 \\ \hline
\vdots & \vdots & \ddots & \vdots \\ \hline
0 & 0 & \vdots & \begin{array}{c|c|c|c} J_{m1} & 0 & \cdots & 0 \\ \hline 0 & J_{m2} & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & J_{ml_m} \end{array}
\end{array}\right]$$

We will denote such a matrix by

$$J_{11} \oplus J_{12} \oplus \ldots \oplus J_{1l_1} \oplus \ldots \oplus J_{m1} \oplus J_{m2} \oplus \ldots \oplus J_{ml_m},$$

or, equivalently, $\bigoplus_{i=1}^{m} \bigoplus_{t=1}^{l_i} J_{it}$ for simplicity. We therefore found the method of finding the **Jordan decomposition** of a matrix:

$$A = P^{-1} \bigoplus_{i=1}^{m} \bigoplus_{t=1}^{l_i} J_{it} P.$$

2. Find Jordan decompositions for matrices

$$\begin{bmatrix} 6 & 2 & -2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix},$$

and

$$\begin{bmatrix} 6 & 2 & 2 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

studied before.

3. Is it always possible to find a Jordan decomposition? What are possible obstacles in the above described process? (hint: think of the characteristic polynomial)

4. **(bonus 50 points)** How Jordan decomposition can be used for solving systems of differential equations? So far we have learned that if

$$A = P^{-1}BP$$

then $e^A = P^{-1}e^BP$. In the special case when it was possible to get B as a diagonal matrix $\bigoplus_{i=1}^{n}[b_i]$, we checked that $e^B = \bigoplus_{i=1}^{n}[e^{b_i}]$. Can we pull off something similar with Jordan decompositions? The following exercises will help you to answer this question.

   (a) Let $A = P^{-1} \bigoplus_{i=1}^{m} \bigoplus_{t=1}^{l_i} J_{it} P$ be the Jordan decomposition of some matrix A. Prove that:

$$e^A = P^{-1} \bigoplus_{i=1}^{m} \bigoplus_{t=1}^{l_i} e^{J_{it}} P.$$

(b) Let, as before

$$J_{it} = \begin{bmatrix} a & 1 & 0 & \cdots & 0 & 0 \\ 0 & a_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & a_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & a_i \end{bmatrix}.$$

Prove that

$$e^{J_{it}} = \begin{bmatrix} e^{a_i} & \frac{e^{a_i}}{1!} & \frac{e^{a_i}}{2!} & \cdots & \frac{e^{a_i}}{(r_{it}-2)!} & \frac{e^{a_i}}{(r_{it}-1)!} \\ 0 & e^{a_i} & \frac{e^{a_i}}{1!} & \cdots & \frac{e^{a_i}}{(r_{it}-3)!} & \frac{e^{a_i}}{(r_{it}-2)!} \\ 0 & 0 & e^{a_i} & \cdots & \frac{e^{a_i}}{(r_{it}-4)!} & \frac{e^{a_i}}{(r_{it}-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & e^{a_i} \end{bmatrix}.$$

(c) Find

$$e^{\begin{bmatrix} 6 & 2 & -2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}}$$

and

$$e^{\begin{bmatrix} 6 & 2 & 2 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}.$$