Practice Questions for the Final

1. Check which of the following functions $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ are linear operators. In case when a given function is linear, find its kernel, image, and matrix (in standard bases):
   (a) $\varphi([x, y, z]) = [x + z, 2x + z, 3x - y + z]$,
   (b) $\varphi([x, y, z]) = [x, y + 1, z + 2]$,
   (c) $\varphi([x, y, z]) = [2x + y, x + z, z]$,
   (d) $\varphi([x, y, z]) = [x - y + z, z, y]$,
   (e) $\varphi([x, y, z]) = [x + z, 2xz, 3x - y + z]$,
   (f) $\varphi([x, y, z, t]) = [x - y + 2t, 2x + 3y + 5z - t, x + z - t]$,
   (g) $\varphi([x, y, z, t]) = [x - y + 2t, 2x - 3y + 5z - t, x - z - t]$,
   (h) $\varphi([x, y, z, t]) = [x + 3y - 2t, x + y + z, 2y + t, y + z]$,
   (i) $\varphi([x, y, z, t]) = [x + 3y - 2t, x + y + z, 2y - 3t, 2x + 4y + z - 2t]$.

2. Show that the vectors $\alpha_1, \ldots, \alpha_n$ form a basis of the vector space V, and find coordinates of the vector β in this basis
   (a) $V = \mathbb{R}^3$, $\alpha_1 = [1, 0, 1]$, $\alpha_2 = [1, 1, 0]$, $\alpha_3 = [1, 1, 1]$, $\beta = [1, 2, 3]$,
   (b) $V = \mathbb{R}^3$, $\alpha_1 = [2, 1, 1]$, $\alpha_2 = [1, 3, 1]$, $\alpha_3 = [1, 1, 4]$, $\beta = [1, 1, 1]$,
   (c) $V = \mathbb{R}^2_2$ (space of 2×2 matrices with real coordinates), $\alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, $\beta = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$,
   (d) $V = \mathbb{C}^2$, $\alpha_1 = [1, i]$, $\alpha_2 = [i, 1]$, $\beta = [1 + i, 2i]$;
   (e) $V = K_3$ (space of polynomials of degree at most 3 and real coefficients), $\alpha_1 = 1$, $\alpha_2 = X - 1$, $\alpha_3 = (X - 1)^2$, $\alpha_4 = (X - 1)^3$, $\beta = X^3 + 2X^2 + X - 1$,
   (f) $V = \mathbb{C}$ (complex numbers viewed as a 2-dimensional vector space over $\mathbb{R}$), $\alpha_1 = 1 + i$, $\alpha_2 = 3i$, $\beta = 2 + 3i$.

3. (a) Linear operator γ has the following matrix in the standard basis:

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & -1 & 2 \\ 2 & 5 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{bmatrix}.$$

   Find its matrix with respect to the following bases:
   (i) $(\varepsilon_1, \varepsilon_3, \varepsilon_2, \varepsilon_4)$ (ε_i denotes the standard vector that has zeros everywhere except for the ith coordinate, where it has 1),
   (ii) $(\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$.

   (b) Find the matrix of a linear operator φ in the basis

$$(\varepsilon_1, \varepsilon_2 + \varepsilon_3, \varepsilon_1 + \varepsilon_2),$$

   if its matrix in the basis
   (i) $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$,
   (ii) $(\varepsilon_1 + \varepsilon_2, \varepsilon_2, \varepsilon_3)$,
   is of the form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

4. Matrix A is the matrix of a linear operator $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ in the standard basis. Find eigenvectors and eigenvalues of φ. Construct (if possible) a basis of $\mathbb{C}^n$ formed by eigenvectors of φ. Find (if possible) a matrix $P \in \mathbb{C}^n$ such that $C^{-1}AC$ is diagonal.

$n = 2$: (a) $A = \begin{bmatrix} 0 & 2 \\ -3 & 5 \end{bmatrix}$, (b) $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$, (c) $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$, (d) $A = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$,

$n = 3$: (e) $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{bmatrix}$, (f) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, (g) $A = \begin{bmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{bmatrix}$.

5. Solve the following systems of differential equations:

(a) $\left\{ \begin{array}{rcrcr} x'(t) & = & & & 2y(t) \\ y'(t) & = & -3x(t) & + & 5y(t) \end{array} \right.$ with initial conditions $x(0) = 1$, $y(0) = 2$.

(b) $\left\{ \begin{array}{rcrcr} x'(t) & = & x(t) & + & 2y(t) \\ y'(t) & = & 2x(t) & - & 2y(t) \end{array} \right.$ with initial conditions $x(0) = 16$, $y(0) = 4$.