

Name:

Perm number:

Midterm – take-home part

1. **(50 points)** Prove the following properties of matrix exponential:

- $e^{aX}e^{bX} = e^{(a+b)X}$
- $e^Xe^{-X} = I$
- if $AB = BA$, then $e^{At}e^{Bt} = e^{(A+B)t}$

2. **(50 points)** In a water purifying plant there are two tanks connected by one tube, each containing 100 gallons of water. Initially the first tank contains 16 pounds of salt, and the second one 4 pounds of salt. The solution of salt and water flows from the second tank to the first one at a rate of 2 gallons per minute. At the same time, fresh water is pumped into the first tank at the rate of 1 gallon per minute, and into the second tank at a rate of 3 gallons per minute, and the solution is pumped out of the first tank at a rate of 3 gallons per minute, and from the second one at a rate of 1 gallon per minute.
- Picture the two tanks with the 5 described above tubes: one pumping the water in the first tank, one pumping it out, same thing for the second tank, and one more tube connecting both tanks. Indicate the corresponding rates of flow.
 - Denote by $x(t)$ and $y(t)$ amount of salt in the first and the second tank, respectively. Write down the equations describing $x'(t)$ and $y'(t)$. You will obtain a system of two differential equations with two variable functions.
 - Write down your system in a matrix form.
 - If A denotes the matrix that appears in your equation, write A as a sum $A = B + C$, where B is a diagonal matrix, and C is what's left of A .
 - Check that $BC = CB$.
 - Use the definition of matrix exponentials to compute e^{Bt} and e^{Ct} .
 - Use what you have learned in Problem 1 to find e^{At} .
 - Now find $x(t)$ and $y(t)$.