Math 5AI Project 6: More on Eigenvalues

If

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is a 2×2 matrix, recall we defined the determinant of A to be

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

If

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is a 3×3 matrix, we define the submatrices

$$A(1|1) = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}, \; A(1|2) = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}, \; A(1|3) = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.$$

We define the determinant of A to be the alternating sum

$$\det(A) = a_{11}\det(A(1|1)) - a_{12}\det(A(1|2)) + a_{13}\det(A(1|3)).$$

1. The $n \times n$ matrix has a determinant which is the generalization of this rule of alternating sums determinant of such submatrices, mutiplied by the entry that is in the row and column eliminated when defining the submatrix. Write out this definition.

2. If A is an $n \times n$ matrix then it can be proved that following conditions are equivalent:
(i) A is invertible, that is, the matrix A^{-1} exists with $A^{-1}A = I_n$.
(ii) $\det(A) \neq 0$.
(iii) The row of A are linearly independent.
(iv) The columns of A are linearly independent.
(v) The rank of A is n.
(vi) The kernel of A is $\{\vec{0}\}$.
Prove the equivalence of all of these in the special case where $n = 2$. We will accept the rest as true for higher n.

If $T : V \to V$ is a linear operator on a vector space, then an λ is an *eigenvalue* of T is a real number λ such that for some nonzero vector $v \in V$ we have $T(v) = \lambda v$. Such a vector v is called an *eigenvector* with eigenvalue λ. We studied a sample of this on problem 4 of the linear transformation sheets.

3. Suppose that $T : V \to V$ is a linear operator. Prove for any λ that the set of eignevectors of T with eigenvalue λ, together with $\vec{0}$, is a vector subspace of V. This subspace is called the *eigenspace* for T with eigenvalue λ.

4. Suppose that A is an $n \times n$ matrix. Form the matrix with polynomial entries $A - xI_n$. We define the *characteristic polynomial* of A to be $C_A(x) := \det(A - xI_n)$.
(i) Using the results from problem 2 show that λ is an eigenvalue of T_A if and only if $C_A(\lambda) = 0$.
(ii) Show that the eigenspace of T_A of eigenvalue λ is the nullspace of the matrix $A - \lambda I_n$.

5. Find the eigenvalues and eigenspaces for the following matrices:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} , \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} , \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Math 5AI, Change of Basis

Definition. Let V be a vector space and let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. An *ordererd basis*, is a such a basis with its elements listed in a specific order, such as $\mathcal{B} = \langle v_1, v_2, \ldots, v_n \rangle$. The reason we care about an order is made clear next. Let $v \in V$. Then since $\mathcal{B}$ is a basis there are unique scalars $a_1, a_2, \ldots, a_n$ such that $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$. We call these scalars the *coordinates of v with respect to $\mathcal{B}$* and we denote this by

$$(v)_{\mathcal{B}} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

1. (a) Let $V = \mathbf{R}^3$ and let $\mathcal{B}_1 = \langle (0,1,0), (0,0,1), (1,0,0) \rangle$. Find the following:

$$\begin{pmatrix} 2 \\ 3 \\ \pi \end{pmatrix}_{\mathcal{B}_1} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}_1}.$$

(b) Find a matrix $P_{\mathcal{B}_1}$ such that for all $v \in V$ that are represented as column vectors, we have $P_{\mathcal{B}_1} \cdot v = (v)_{\mathcal{B}_1}$.
(c) Let $V = \mathbf{R}^3$ and let $\mathcal{B}_2 = \langle (1,1,0), (0,1,1), (1,0,1) \rangle$. Find the following:

$$\begin{pmatrix} 2 \\ 3 \\ \pi \end{pmatrix}_{\mathcal{B}_2} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}_2}.$$

(d) Find a matrix $P_{\mathcal{B}_2}$ such that for all $v \in V$ that are represented as column vectors, we have $P_{\mathcal{B}_2} \cdot v = (v)_{\mathcal{B}_2}$.
(e) Find a matrix $Q_{\mathcal{B}_2}$ such that for all $v \in V$ that are represented as column vectors, we have $Q_{\mathcal{B}_2} \cdot (v)_{\mathcal{B}_2} = v$.
(f) Explain how, for any basis $\mathcal{B}$ of $V = \mathbf{R}^3$ to find matrices $P_{\mathcal{B}}$ and $Q_{\mathcal{B}}$ such that for all $v \in V$ that are represented as column vectors, we have $P_{\mathcal{B}} \cdot v = (v)_{\mathcal{B}}$ and $Q_{\mathcal{B}} \cdot (v)_{\mathcal{B}} = v$.

Definiiton. Suppose that $T : V \to V$ is a linear operator and that $\mathcal{B} = \langle v_1, v_2, \ldots, v_n \rangle$ is an ordered basis for V. Then the matrix $[T]_{\mathcal{B}}$ is the unique $n \times n$ matrix with the property that for $v \in V$ we have

$$[T]_{\mathcal{B}} \cdot (v)_{\mathcal{B}} = (T(v))_{\mathcal{B}}$$

2. (a) Look closely at the definition of $[T]_{\mathcal{B}}$ just given. Be sure you know what is going on here.
(b) Let $V = \mathbf{R}^3$ and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the bases given in problem 1 above. Let $T = T_A$ where A is the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Find each of the matrices $[T]_{\mathcal{B}_1}$ and $[T]_{\mathcal{B}_2}$ for this T. In doing this: *Figure out how to use the matrices $P_{\mathcal{B}_i}$ and $Q_{\mathcal{B}_i}$ used in problem 1. Don't try to find $[T]_{\mathcal{B}_i}$ by starting at the beginning.*

3. (a) Earlier (in problem 5 of the eignevalue sheet) you found the eigenvalues and eigenspaces of the following matrix:

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Take an ordered basis of eigenvectors for $V = \mathbf{R}^3$ and find the matrix of T_B with respect to this basis.

4. Suppose that $T : V \to V$ is a linear operator and suppose that v_1, v_2 and v_3 are eigenvectors with three *distinct* eigenvalues. Prove that v_1, v_2 and v_3 are linearly independent. Hint: As a warm-up do the case of two vectors first.

Math 5AI, Linear Algebra Quiz

Consider the following matrix.

$$A = \begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}$$

The reduced row-echelon form of A is

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Using A and R answer 1 and 2 below. You do not need to compute very much since the row reduction of A has been provided!

1. Give a basis for the set of solutions to the system of homogeneous equations $A\vec{x} = \vec{0}$. (Here, $\vec{x}$ is a column of unknowns of the appropriate size and $\vec{0}$ is a column of zeros of the appropriate size.)

2. Let $T_A : R^n \to R^m$ be the linear transformation defined by A.
(i) What are the correct values of m and n for this statement to make sense?
(ii) Find bases for both the image $\mathrm{im}(T_A)$ and the kernel $\ker(T_A)$.

3. (i) Find the eigenvalue and eigenspaces of the operator T_B if

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(ii) Give bases for these eigenspaces.

4. (Bonus Problem) (i) Suppose that $T : V \to V$ is a linear operator and suppose that v_1, v_2 and v_3 are linearly independent in V. Suppose also that $\ker(T) = \{\vec{0}\}$. Show that $T(v_1)$, $T(v_2)$ and $T(v_3)$ are also linearly independent in V. Hint: Use the linearity property of T.
(ii) Give an example that shows the conclusion of (i) fails if $\ker(T) \neq \{\vec{0}\}$.