Math 5AI, Linear Transformations

Note. We are now going to increase the abstraction in this course by using the terminology of vector spaces, bases, linear independence in everything we do. We will study a particular type of function between vector spaces, called a *linear transformation*. They are critical to understanding to most of the material that remains in 5AI and in 5B. So be ready to dig in and stick with it even if the abstraction feels uncomfortable at times. We will get through this hurdle.

For us, a *vector space* is a set V of real valued functions for which every sum of two elements and every multiple of an element remains in V. Formally, this means if $f_1,\ f_2 \in V$ then $f_1 + f_2 \in V$ and if $f \in V$ and $r \in \mathbf{R}$ then $rF \in V$.

We recall that if $f_1, f_2, \dots f_n \in V$ and if $r_1, r_2, \dots r_n \in \mathbf{R}$ then $r_1 f_1 + r_2 f_2 + \cdots + r_n f_n$ is called a *linear combination* of $f_1, f_2, \dots f_n$. The set of linear combinations of $f_1, f_2, \dots f_n$ is called their *span* and is denoted $\text{Span}\{f_1, f_2, \dots f_n\}$. We recall that $f_1, f_2, \dots f_n \in V$ are called *linearly independent* if the only case where $r_1 f_1 + r_2 f_2 + \cdots + f r_n f_n = 0$ is when $r_1 = 0$, $r_2 = 0$,$\dots$, $r_n = 0$. Finally, a *basis* of V is a set $\{f_1, f_2, \dots f_n\}$ of elements that are linearly independent and span V. If V has a basis of n elements, it is called n-dimensional.

1. We denote by $\mathbf{P}^3$ the vector space of all polynomials of degree at most three. In other words, $\mathbf{P}^3 = \{ax^3 + bx^2 + cx + d\}$ where $a, b, c, d \in \mathbf{R}$. Find two different bases for $\mathbf{P}^3$. What is its dimension?

2. Consider the following function $T : \mathbf{P}^3 \to \mathbf{P}^3$ given by

$$T(f) = (x^2 + 2x + 1)f'' - 3(x + 1)f' + 3f.$$

Explain why T is linear. Recall that T being linear means that $T(rf) = rT(f)$ and $T(f_1 + f_2) = T(f_1) + T(f_2)$. Do not give a long calculation–you should be able to give a proof discussing what you know about derivatives.

3. The *kernel* of T is $\ker(T) := \{f \in \mathbf{P}^3 \mid T(f) = 0\}$. Describe which functions are in $\ker(T)$. Show that $\ker(T)$ is a vector space and find a basis for $\ker(T)$.

4. The *image* of T is $\text{im}(T) := \{T(f) \in \mathbf{P}^3 \mid f \in \mathbf{P}^3\}$. Describe which functions are in $\text{im}(T)$. Show that $\text{im}(T)$ is a vector space and find a basis for $\text{im}(T)$.

5. Find a 4×4 matrix A that calculates T in the following way: Suppose that $T(ax^3 + bx^2 + cx + d) = ex^3 + fx^2 + gx + h$. Then

$$A \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}$$

6. Rework each of the questions in problems 3., 4., 5. for the linear transformation $D : \mathbf{P}^3 \to \mathbf{P}^3$ given by $D(f) = f'$.

7. Rework each of questions in problems 3., 4., 5. for the linear transformation $L(ax^3 + bx^2 + cx + d) = ex^3 + fx^2 + gx + h$ given by

$$B \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Math 5AI, Linear Transformations, continued

1. Recall the linear operator $T : \mathbf{P}^3 \to \mathbf{P}^3$ given by

$$T(f) = (x^2 + 2x + 1)f'' - 3(x + 1)f' + 3f.$$

We saw earlier that if we restrict or attention to the coefficients of a polynomial $f = ax^3 + bx^2 + cx + d$ then T is given by the matrix multiplication

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 6 & -2 & 0 & 0 \\ 0 & 2 & -3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Let A dentote the 4×4 matrix in this expression.
(a) Explain why the span of the columns of A can be thought of as giving the image of T.
(b) Suppose you solve the system of equations in four variables, z_1, z_2, z_3, z_4, $A\vec{z} = \vec{0}$. What do the solutions to this equation have to do with differential equations?
(c) Find all solutions to the linear systems

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 6 & -2 & 0 & 0 \\ 0 & 2 & -3 & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 6 & -2 & 0 & 0 \\ 0 & 2 & -3 & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}.$$

(d) Explain what your solutions to (c) mean in terms of differential equations.
(e) Important: Expand you discussion in (d) to include the existence and uniqueness theorem for these ODE.

2. We are now going to set differential equations aside and work with matrices. If A is a $m \times n$ matrix we denote by $\vec{v} \in \mathbf{R}^n$ an element *written as a column matrix* and then we obtain a linear transformation $T_A : \mathbf{R}^n \to \mathbf{R}^m$ defined by $T_A(\vec{v}) = A \cdot \vec{v}$. In other words, T_A is the function obtained by multiplying columns on the left by the matrix A. We call T_A the *linear transformtion* determined by the matrix A.

For each of the following matrices, find bases the kernel and images of the linear transformations they determine. For this you will need to know for which m and n they define a function $\mathbf{R}^n \to \mathbf{R}^m$.

$$\begin{pmatrix} 3 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 \\ 2 & 3 \\ 4 & 6 \\ 6 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

3. If A is a matrix, the dimension of the kenel of T_A is called the *nullity* of A and the dimension of the image of $T_(A)$ is called the rank of A.
(a) Look at the reduced row-echelon forms of each matrix in problem 2 above and discuss the rank and nullity of each in terms of the reduced row-echelon forms. (You determined the rank and nullity of each of these in problem 2.)
(b) Formulate a conjecture about the relationship between the rank and nullity of an arbitrary matrix.
(c) Prove your conjecture.

4. Consider the matrix

$$F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and the column vector} \quad V = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(a) Find FV, F^2V, F^3V and describe $F^{n+1}V$ in terms of F^nV. Do you recognize the entries of these matrices?
(b) Find a column vector E for which $FE = \lambda E$ for some real number λ. (You will have to solve a quadratic equation to find the constant λ. After finding λ you can find E.) The constant λ is called an *eigenvalue* of A and the vector E is called an *eigenvector.*
(c) Calulate $F^{20}V$ using a calculator and calculate the ratio of the entries of the result (the top entry divided by the bottom entry). Express you eigenvalue(s) from part (b) as a decimals and compare them to this ratio.
(d) Let

$$W = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Calulate $F^{20}W$ using a calculator and calculate the ratio of the entries of the result (the top entry divided by the bottom entry). What do you notice? What do you think this means geometrically?

5. For any real number θ with $0 \leq \theta \leq 2\pi$ we consider the matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(a) Calculate $\det(R_\theta)$.
(b) Find the eigenvalues of R_θ if any.
(c) Explain your answer to (b) geometrically by describing the linear transformation $T_{R_\theta} : \mathbf{R}^2 \to \mathbf{R}^2$.