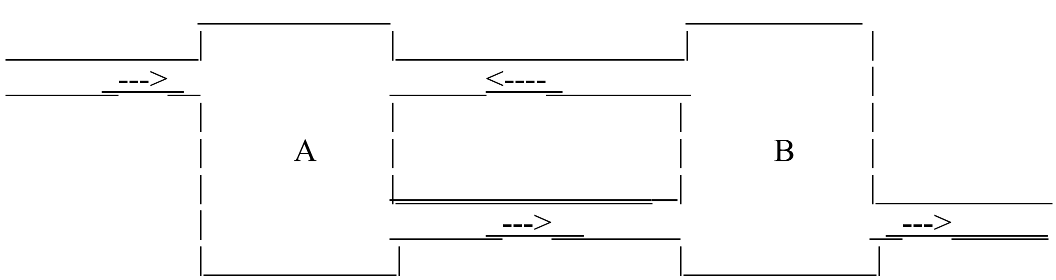
Math 5AI   Mixing Salt Water

In this project you will first create a system of ODE that describe a model for mixing different salt water solutions. This situation is much like our earlier projects. However, instead of studying methods for approximating solutions like last quarter, you will develop exact symbolic solutions. There are many applications of this type of problem -- for example, variation of salt concentrations under tidal or fresh water source changes is important in marine ecology -- and in fact this type of modeling has played a key role in the California water debates over the past three decades (for example the north vs. south battle over the "Peripheral Canal".)

1. Two 100 gallon containers (labeled A and B) are connected as pictured.



Fresh water flows into A at a rate of four gallons/minute and flows out of B at the same rate. Water flows from A to B at a rate of 7 gallons/minute and from B to A at 3 gallons per minute (in the directions pictured). Assume that there is no air in these containers and that the salt solutions mix "instantly" after the solution flows in/out (and that no mixing takes place across the pipes.) Assume that at time 0, there are 16 lbs of salt in container A and that there are 4 lbs of salt in container B. Let $x(t)$ denote the lbs of salt in container A at time t, and let $y(t)$ denote the lbs of salt in container B at time t. Find a system of differential equations that describe $x(t)$ and $y(t)$.
NOTE: If the written description isn't clear, we will talk about it. The best description of the behavior of the system is obtained by writing down a system of ODE!

2. Who remembers matrix multiplication? Write your system of ODE's obtained in the first part as a single matrix multiplication, using a 2 by 2 matrix and the "pair" of functions $x(t)$, $y(t)$ as a column.

3. Quick review: How do you find all solutions to the diffyQ $y'(t) = ky(t)$ for k a constant? What is necessary to specify a unique solution?

4. Wouldn't it be nice if one could write e^A for a square matrix A (with real entries) and obtain another square matrix. Well, in fact you can! Recall the infinite series that converges to e^x for real values x ? Show how you can use this formula to define e^A !! This will force you to think about convergence of infinite series. OK - that will be part of our class work.

Mixing Salt Water, Continued

5. Now consider a matrix of functions e^{At} . Show that it converges for all real t . Suppose that one defines the derivative of a matrix of functions to be the matrix of functions whose entries are the derivatives of the original functions. What do you suppose the derivative, e^{At} is? Stick to the 2 by 2 case.

6. How does what you have learned help with the original problem?

7. It turns out to be very important to figure out easier ways to compute matrices of functions of the form e^{At} . For this, check the following.
(a) Suppose that $A = PBP^{-1}$. Then show that $e^A = P(e^B)P^{-1}$.
(b) If B is diagonal, find e^B (quickly!)

8. In class we will find out how to express your matrix A from part 2 of this project in the form $A = PBP^{-1}$ for a diagonal matrix B . Now use this to find a symbolic solution to the original problem!

9. At what time is the salt in both tanks equal?

10. Does the salt ever get diluted to less than one lb. per tank?

Bill: Answer in source was WRONG!!!! Must have flow of 7 gallons A to B for stable system.

$$A = \begin{pmatrix} -7/100 & 3/100 \\ 7/100 & -7/100 \end{pmatrix}$$

which has eigenvalues $7 \pm \sqrt{21}$

and corresponding eigenvectors $\begin{pmatrix} 3 \\ \sqrt{21} \end{pmatrix}$ $\begin{pmatrix} 3 \\ -\sqrt{21} \end{pmatrix}$.

This means that if P is $\begin{pmatrix} 3 & 3 \\ \sqrt{21} & -\sqrt{21} \end{pmatrix}$

and $P^{-1} = (-1/6\sqrt{21}) \begin{pmatrix} -\sqrt{21} & -3 \\ -\sqrt{21} & 3 \end{pmatrix}$

then $D = P^{-1}AP$ is diagonal and students can go from here.