Math 5AI Third Project Getting back to Linear Algebra: The Wronskian

1. An important result about second order homogenous linear ODE's

$$y'' + p(x)y' + q(x)y = g(x)$$

is that, assuming $p(x)$, $q(x)$, and $g(x)$ are continuous on an interval, there is a unique solution defined on that interval that satisfies initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$. We used power series to see that this result is plausible although what we did is not quite a proof.
(a) Show that this means the homogenous equation $y'' + p(x)y' + q(x)y = 0$ has a 2-dimensional space of solutions. (In other words, show that there are two linearly independent solutions for which em every solution is a linear combination. (Assume that $p(x)$ and $q(x)$ satisfy the continuity condition.)
(b) What can you say about the solutions to $y''+p(x)y'+q(x)y = 0$ for which $y(0) = 0$ and $y'(0) = 0$? (Assume that $p(x)$ and $q(x)$ satisfy the continuity condition.)

Let y_1 and y_2 be two functions. Then we define the *Wronskian* of y_1 and y_2 to be

$$W(y_1, y_2) = y_1y_2' - y_2y_1' := \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

2. One use of the Wronskian is to determine if a pair of solutions to an ODE are linearly independent.
(a) Suppose that every solution to $y'' + p(x)y' + q(x)y = 0$ can be expressed as a linear combination of y_1 and y_2. Show that the function $W(y_1, y_2)$ is never 0. (Hint: Suppose the Wronskian was 0 at a point x_0. What would it mean that you can find solutions to the ODE of the form $y = Ay_1 + By_2$ for all possible initial conditions $y(c) = d$ and $y'(c) = e$?)
(b) Conversely, if the Wronskian of never vanishes y_1 and y_2, show that they must be a basis for the vector space of solutions to the ODE.
(c) Find the Wronskian of a pair of solutions to $y'' + y = 0$.
(d) Find the Wronskian of a pair of solutions to $y'' - y = 0$.

3. *Calculating the Wronskian.* In general, it may not be clear whether or not the Wronskian vanishes at a point or not. But you will show next that the Wronskian of two solutions to a linear ODE is either never zero or is always zero. For this, suppose that y_1 and y_2 are solutions to $y'' + p(x)y' + q(x)y = 0$. Then we have

$$\begin{aligned} y_1'' + p(x)y_1' + q(x)y_1 &= 0 \\ y_2'' + p(x)y_2' + q(x)y_2 &= 0 \end{aligned}$$

Multiply the first equation by $-y_2$ and the second equation by y_1 and add to obtain a new equation. See that this equation is an ordinary differential equation in the function $W(y_1, y_2)$ by factoring out expressions equal to $W(y_1, y_2)$ and to $W(y_1, y_2)'$ in this expression. Now solve this ODE using an integratng factor. You will find why the Wronskian is identically zero or never zero.

4. *The Viewpoint of Linear Oerators.* (a) Consider the expression

$$T(y) = y'' + p(x)y' + q(x)y$$

This is what we call a *linear operator* because the function $L(y)$ operates linearly on functions. This means that $L(y_1 + y_2) = L(y_1) + L(y_2)$ for all twice differentiable functions y_1 and y_2 and that $L(ky) = kL(y)$ for all all twice differentiable functions y and real numbers k. Explain this.
(b) Now consiser the operator

$$T(y) = y'' + 2xy' + 2y$$

Calculate, $T(1)$, $T(x)$, $T(x^2)$, $T(e^x)$, $T(e^{-x^2}$.
(c) Find a solution to $T(y) = x^2$.
(d) Find a solution to $T(y) = x^2$ for which $y(0) = 0$.
(e) Are there other solutions to $T(y) = x^2$ for which $y(0) = 0$? What would you have to do to find another?