Math 5AI Second Project: Existence of Solutions to ODEs: Power Series

1. Whenver $|r| < 1$, I claim that the infinite sum $1 + r + r^2 + r^3 + \cdots$ makes sense as a limit of partial sums. For this we multiply these partial sums by by $(1 - r)$ algebraically to see what happens. Explain this.

2. Now look at the expression $E(x) = 1 + x + x^2/2 + x^3/3! + x^4/4! + \cdots$. Use the fact that for any real number x it must happen that $\mid x^n \mid < n!$ for for large enough n to see that $E(x)$ defines a function. Assuming that you can take the derivative of $E(x)$ by pretending it is an infinitely long polynomial show that $E(x)$ is a solution to the ODE $y' - y = 0$ with $E(0) = 1$. Later, for homework you can try to prove this assertion about the derivative of $E(x)$, but not now.

3. Take five minutes and look for a solution to Airy's Equation: $y'' - xy = 0$. If you find a solution write it on the board and share it with the class. After five minutes go on to problem 4.

4. (a) Imagine that Airy's solution has solutions as a power series. What could these power series be? Find out. What about initial conditions in your solution? Don't forget about these and be sure you can discuss them in your write-up.
(b) Use the ideas from problems 1 and 2 to see this power series does define a function.

5. Use the ideas in 4 to give power series solutions to $y'' + y = 0$.

6. Find the first three terms of two linearly independent power series in x to the equation $y'' + (\sin x)y = 0$.

Math 5AI Short Quiz

1. Find all solutions to $y'' + y = x^2$.

2. Consider the ODE

$$y''' - y'' + y' - y = 0.$$

(a) What are all of its solutions?
(b) What are all of its solutions for which $y(0) = 0$, $y'(0) = 1$, and $y''(0) = 0$?
(b) What are all of its solutions for which $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 0$?

3. Show how to derive a power series solution to $y'' + 2y = 0$ for which $y(0) = 1$ and $y'(0) = 0$. List terms up to x^4.