Math 5AI First Project: Minimizing the Bounce

In this investigation your task is to help design a car so that you don't get seasick or break your rear-end when you hit a pothole.

1. Your building a car, which when completed will weigh $4,000$ pounds. You have installed a simple coil spring suspension (not too common these days, actually, but they are cheap) and an engineering friend of yours suggests that you might want to add shock absorbers. This first problem addresses whether or not this would be a good idea and why.

With the car stationary and sitting still you note that if you add 500 pounds it dips 3 inches as the spring compresses due to the weight. In order to think about what happens when you hit a pothole, we set up a height function $h(t)$ that will measure the displacement of the car from its normal height as a function of time. We normalize this function so that $h(t) = 0$ when the car is stationary, and so that $h(t) = d$ will mean the springs have streched d feet. When $d < 0$ the springs have compressed $-d$ feet. In particular $h(t) = -1.5$ is bad because you bottom out since you only have 18 inches of clearance.

Basic Physics tells us $F = ma$ (force = mass times acceleration) and Hookes Law tells us that the force of a spring is proportional to the distance stretched (or compressed), in other words the force of the spring is $k \cdot d$ where d is the distance compressed and k is a constant. Since weight is mg where m is mass and g is the acceleration of gravity, we obtain $m = w/g = 4000\text{lb}/(32\text{ft}/\text{sec}^2) = 125\text{lb}\cdot\text{sec}^2/\text{ft}$ and since 500 lbs compresses the springs 1/4 ft, we find that the force of the springs are $F_s = 2000 \cdot d$, where d is the compressed distance.

Now, when you hit a pothole the car is given a downward jolt, and the springs compress, and we need to model the resulting up and down motion using the ODE

$$m \cdot h''(t) = -k \cdot h(t)$$

since the force of the spring needs to be match the force resulting from the acceleration of the car. Your problem is to determine what happens when you hit a pothole and the spring is compressed to $d = -.5$ ft, that is you solve this ODE for $h(t)$ with one initital condition being $h(0) = -.5$. Do you think a shock absorber is a good idea? Why?

2. OK, maybe you aren't sure if you want a shock absorber, but you decide to investigate it anyway. You are told by your engineer friend that a basic shock absorber exerts a force proportional to the speed of motion of the rod through the absorber (this in fact may not be accurate, but it will be used in this model anyway). We

will use c for this constant. What this means is that the ODE considered above now becomes

$$m \cdot h''(t) + c \cdot h'(t) + k \cdot h(t) = 0$$

There are some ready made shock absorbers with $c = 500$ or $c = 2000$ you can use if you like, or you might choose to specify a different c value. What do you choose? Investigate these possibilities. You may need to consider the following digression problem.

Digression. You may find it useful to think about complex numbers when considering the solutions to 2.
(a) For this, you will want to start with Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Derive Euler's formula from the Taylor series at 0 for e^x, $\cos x$, and $\sin x$.
(b) Now, as an exercise, find two linearly independent *complex* solutions for $y'' + y' + y = 0$.
(c) Use linear combinations of your solutions to (b) to find two linearly independent *real* solutions for $y'' + y' + y = 0$.
(d) Take what you have learned here to go back to problem 2.

3. Now develop your ideas from problem 2 more fully and write up a general rule for solving second order homogeneous constant coefficient linear ODE's. When you have it nailed down, extend your ideas to even higher order homogeneous constant coefficient linear ODE's. Give some examples.

4. Develop a general theory for solutions to homogeneous Euler-Cauchy equations analogous to what you have done for problem 3. These equations are of the form

$$at^2y'' + bty' + cy = 0$$

and you should be considering possible solutions of the form $y = kt^r$ or possibly $y = kt^r \ln t$ when you have repeated roots.

Write-ups on problems 3. and 4. will be due on a date to be specified in class. We will complete our discussion of the shock absorber then too and move on to the problems of driving on a bumpy road.

Math 5AI   Minimizing the Bounce (Continued): Hitting the Dirt Road

1. You decide to take your car on a spin along a county road which is all dirt and a bit rough. Here is an ODE to help you figure out what this might be like. You will assume that the bumps in the road are wavy and exert a regular oscillting force as you drive.
(a) First we will work this out without shock absorbers. Recall our original "smooth road" equation

$$m \cdot h''(t) + k \cdot h(t) = 0$$

with $m = 125$ and $k = 2,000$. This was fine for a smooth road and we could model one bump. But now we assume the force of the road is given by $F = .3\cos(3t)$. Solve the ODE

$$m \cdot h''(t) + k \cdot h(t) = .3\cos(3t)$$

with initial conditions given by hitting the 6 inch pothole. You will need some solution to this ODE to get started. For this use undetermined coefficients: Find A and B so that $h(t) = A\sin(3t) + B\cos(3t)$ is a solution. Then use your homogeneous information to fix this to meet the initial conditions. Use a graphing calculator or other technology to plot your solution.
(b) There is a different road whose bumps are given by $F = .1\cos(4t)$. What is the difference between this road and the road considered in part (a). How do you think this different road will affect your driving (again without a shock absorber)? Write out your conjecture first and then solve the problem. Compare your answer to what you found in (a) using a calculator or other technology to plot your solution. What do you notice.
(c) Go back to your solution given in (a) and use the identity

$$\cos u - \cos v = -2\sin\left(\frac{u-v}{2}\right)\sin\left(\frac{u+v}{2}\right)$$

to reexpress the solution as a product of sine functions. Use this expression to discuss the graph you obtained in (a) qualitatively.

2. You have learned to solve general second order homogeneous, constant coefficient ODE's and you have in the above learned to use the method of *undetermined coefficients* to find inhomogeneous solutions, whereby you make intelligent guesses about what the general form of a solution could be and then solve for constants. Now you get to try your hand a solving more ODE's in this way. Find the general solutions to the following.
(a) $y' = 1$
(b) $y'' = 1$

(c) $y'' + 4y' = t$
(d) $y'' + y' - 2y = 6e^t$
(e) $y'' - 3y' = 2y = e^t \sin t$
(f) $y'' - y' - 6y = e^{-t}$
(g) $y'' - y' - 6y = \cosh t$

3. As a project, go back to problem 1 (a) with $F = 500\cos(3t)$ and and rework with the shock absorber you developed earlier. How does you shock absorber work now? We will discuss this problem later when class members have solutions to share.