Math 3CI Basis and Dimension

Definniton. Suppose that V is a vector space. This means it could be a function space or a subset of $\mathbf{R}^n$ that is closed under addition and scalar multiplication. A *basis* for V is a spanning set of minimal size.

It turns out, that for any vector space the number of elements in any two basis is always the same. We will prove this in a bit. This number is called the *dimension* of the vector space.

We have seen that if you solve a homogeneous linear ODE, then the set of solutions is a function space, and if you solve a homogeneous system of linear equations, the set of solutions is a vector space. When we solve non-homogeneous linear ODEs or nonhomogeneous systems of linear equations, the set of solutions is obtained from one particular solution by adding all the solutions in the vector space of the associated homogeneous equations.

1. (a) Find a basis for the following homogeneous system of linear equations

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 0 \\ x & + & y & + & z & = & 0 \\ 3x & + & 5y & + & 7z & = & 0 \end{array}$$

(b) Use your basis to find all solutions to

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 6 \\ x & + & y & + & z & = & 3 \\ 3x & + & 5y & + & 7z & = & 15 \end{array}$$

(c) Use your basis to find all solutions to

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 6 \\ x & + & y & + & z & = & 3 \\ 3x & + & 5y & + & 7z & = & 10 \end{array}$$

(d) Find a basis for the set of solutions to thes following homogenous system

$$\begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(e) Use your basis to find all solutions to

$$\begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \\ v \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

where c_1, c_2, and c_3 are arbitrary constants.
(f) Compare what happened in (e) to what happened in (b) and (c).

2. Think about what happened in 1 (d) above.
(a) Find a system of equations whose soluton is the span of the set of vectors:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

(b) Find a system of equations whose soluton is the span the of set of vectors:

$$\begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

3. (a) Show that a mininmal spanning set of a vector space consists of linearly independent vectors.
(b) Take a set of vectors, such as those in problem 2. Use them as the *rows* of a matrix. Show that "row operations" don't change the span of the rows.
(c) Show that the nonzero rows of a reduced echelon matrix are linearly independent.
(d) For any system of homogeneous linear equations explain why there is a unique reduced row-echelon matrix R such that the set of solutions to $R\vec{x} = \vec{0}$ is the same as that of the original system. (Hint: compare the homogeneous solution sets to two different homogeneous reduced row-echelon systems.)