Math 3CI Basic Linear Algebra Ideas

We studied linear differential equations earlier and we noted that if one has a homogeneous linear differential equation

$$(*) \qquad y^{(n)} + f_{n-1}y^{(n-1)} + \cdots + f_2y'' + f_1y' + f_0y = 0$$

and if g and h are solutions to $(*)$, then for all $k \in \mathbf{R}$ both $g+h$ and kg are solutions to $(*)$. This motivates the following definition.

Definition. Suppose V is a set of functions from $\mathbf{R}$ to $\mathbf{R}$ and suppose that whenever if g and h are elements of V, then for all $k \in \mathbf{R}$ both $g+h$ and kg are elements of V. Then V is a *vector space*, or since its elements are functions, we sometimes call it a *function space.*

So we note, in particular, that that the set of solutions to a homogeneous linear differential equation is a function space. We are going to study function spaces for a while. For this we need a few more ideas.

Definition. (i) Suppose u is a function. Then the set of all multiples of u, $\mathrm{Span}\{u\} := \{cu \mid c \in \mathbf{R}\}$ is call the set of *linear combinations* of u.
(ii) Suppose u and v are functions. Then the set of all sums, $\mathrm{Span}\{u, v\} := \{c_1u + c_2v \mid c_1, c_2 \in \mathbf{R}\}$ is called the set of *linear combinations* of u and v.
(iii) Suppose u, v, w are functions. Then the set of all sums, $\mathrm{Span}\{u, v, w\} := \{c_1u + c_2v + c_3w \mid c_1, c_2, c_3 \in \mathbf{R}\}$ is called the set of *linear combinations* of u, v and w.
(iv) I bet you can figure out what a linear combination of four or more functions is. We often call the set of linear combinations of a set of functions, the *span* of those functions, as the notation suggests.

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1. (a) Suppose that u, v, w are functions none of which are zero. If $u+v+w = 0$ show that

$$\mathrm{Span}\{u, v, w\} = \mathrm{Span}\{u, v\} = \mathrm{Span}\{u, w\} = \mathrm{Span}\{v, w\}$$

(b) Suppose that u, v, w are functions none of which are zero. If $3u + 2v - 17w = 0$ show that

$$\mathrm{Span}\{u, v, w\} = \mathrm{Span}\{u, v\} = \mathrm{Span}\{u, w\} = \mathrm{Span}\{v, w\}$$

(c) Suppose that u, v, w are functions none of which are zero. If $au+bv+cw=0$ and each of a, b, c are nonzero, show that

$$\mathrm{Span}\{u,v,w\}=\mathrm{Span}\{u,v\}=\mathrm{Span}\{u,w\}=\mathrm{Span}\{v,w\}$$

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Definition. If v_1, v_2, v_3,..., v_n are n functions, we say that these functions are *linearly dependent* if there exist real constants a_1, a_2, a_3,..., a_n, not all zero, with $a_1v_1+a_2v_2+a_3v_3+\cdots+a_nv_n=0$.

2. Which of the following collections of functions are linearly dependent?
(a) 1, x, x^2, x^3+x^2+x+1.
(b) 1, x, x^2, x^3, x^3+x^2+x+1.
(c) $x+1$, x^2+1, x^3+1, x^3+x^2+x+1.
(d) $x+1$, x^2+1, x^3+1, x^3+x^2+x+1, -1.

3. Generalize the ideas in problem 1 above and show that if a collection of n functions v_1, v_2, v_3,..., v_n are *linearly dependent* then thier span is the same as the span of fewer of these functions.

4. Generalize the ideas in problem 2 above and show that if a collection of n polynomials each have different degree, then they are *linearly independent* (which means they are not linearly dependent.)

5. Suppose that u, v, w are functions and that $u=v+cw$ for some $c\in\mathbf{R}$. Show that for any functions y_1, y_2, y_3,..., y_n that

$$\mathrm{Span}\{v,w,y_1,y_c,\ldots,y_n\}=\mathrm{Span}\{u,w,y_1,y_c,\ldots,y_n\}.$$

6. Find a set of polynomials of different degrees which has the same span as

$$x^3+x^2+x+1,\ x^3+2x^2+3x+4,\ x^3+2x^2+2x+2,\ x^3+x^2+2x+3.$$

7. Suppose you have a finite set of polynomials. Can you always find a set of polynomials of different degrees with the same span? Explain your thinking carefully.

Math 3CI More Basic Linear Algebra Ideas

1. Consider the span of all functions of the form: e^x, e^{2x}, e^{3x},..., $\sin x$, $\sin 2x$, $\sin 3x$, ..., $\cos x$, $\cos 2x$, $\cos 3x$, ... In this span search for solutions to the following *constant coefficient homogeneous linear* ODEs. This is a string of problems where you should use the ideas in the previous problems to help you with the next problem! They should not be too laborious.
(a) $y' + ky = 0$, where k is a positive or negative integer.
(b) $y'' - 7y' + 12y = 0$
(c) $y'' + 6y' + 9y = 0$
(d) $y'' + k^2y = 0$, where k is an integer.
(e) $y'' - k^2y = 0$, where k is an integer.

2. (a) Describe all real values of x, y, and z which are solutions to

$$\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 0 \\ x & + & y & + & z & = & 0 \\ x & + & 4y & + & 7z & = & 0 \end{array}$$

(b) Describe all real values of x, y, and z which are solutions to

$$\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 0 \\ x & + & 4y & + & 7z & = & 0 \end{array}$$

3. (a) Describe all real values of x, y, z, and w which are solutions to

$$\begin{array}{ccccccccc} x & + & 2y & + & 3z & + & 4w & = & 0 \\ x & + & y & + & z & + & w & = & 0 \\ x & + & 4y & + & 7z & + & 10w & = & 0 \end{array}$$

(b) Describe all real values of x, y, and z which are solutions to

$$\begin{array}{ccccccccc} x & + & 2y & + & 3z & + & 4w & = & 0 \\ x & + & 4y & + & 7z & + & 10w & = & 0 \end{array}$$

4. (a) Describe all real values of x, y, z, and w which are solutions to

$$\begin{array}{ccccccccc} x & + & 2y & + & 3z & + & 4w & = & 0 \\ 2x & + & 4y & + & 6z & + & 8w & = & 0 \end{array}$$

(b) Describe all real values of x, y, and z which are solutions to

$$x + 2y + 3z + 4w = 0$$

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Note: If you didn't attend class last Wednesday, you should work on the sheet for that day outside of class. You should be able to tackle todays problems before completing that sheet.

We are now going to consider vector subspaces of $\mathbf{R}^n$ where $n = 1, 2, 3, \ldots$. For us, $\mathbf{R}^n$ is the set of n-tuples of real numbers: $\mathbf{R}^n := \{(r_1, r_2, \ldots, r_n) \mid r_1, r_2, \ldots, r_n \in \mathbf{R}\}$. But for reasons you will see shortly, sometimes it is more convenient to represent elements of $\mathbf{R}^n$ as columns:

$$\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$$

We will use either rows or columns, depending upon which is convenient at the time.

A vector subspace of $\mathbf{R}^n$ is a subset closed under addition and multipication by real numbers. In $\mathbf{R}^2$, for example, the vector subspaces are of three types: (i) the zero subspace $\{(0,0)\}$, (ii) lines through the origin in $\mathbf{R}^2$, and (iii) all of $\mathbf{R}^2$.

1. What are the types of subspace of $\mathbf{R}^3$. This is a two-minute question.

2. Below are four systems of equations. They are related. Describe all real values of x, y, and z which are their solutions.

System 2.1:

$$x + y - z = 0$$

System 2.2:

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 0 \\ x & + & y & + & z & = & 0 \end{array}$$

System 2.3:

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 0 \\ x & + & y & + & z & = & 0 \\ x & - & y & + & z & = & 0 \end{array}$$

System 2.4:

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 0 \\ x & + & y & + & z & = & 0 \\ x & - & y & + & z & = & 0 \\ 3x & + & y & + & z & = & 0 \end{array}$$

Are any of these solution sets vector subspaces of $\mathbf{R}^3$?

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3. Below are four systems of equations. They are related. Describe all real values of x, y, and z which are their solutions.

System 3.1:

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 1 \end{array}$$

System 3.2:

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 1 \\ x & + & y & + & z & = & 2 \end{array}$$

System 3.3:

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 1 \\ x & + & y & + & z & = & 2 \\ x & - & y & + & z & = & 3 \end{array}$$

System 3.4:

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 1 \\ x & + & y & + & z & = & 2 \\ x & - & y & + & z & = & 3 \\ 3x & + & y & + & z & = & 4 \end{array}$$

Are any of these solution sets vector subspaces of $\mathbf{R}^3$? How are the solution sets to these systems related to the solution sets of problem 2?