Math 3CI Even More about solving DiffyQ Symbolicallly Part IV

In these problems you are pushed to develop some more symbolic techniques for solving ODE's that extends the Chain-rule based techniques that gave implicit descriptions of the functions. The differential equations that can be solved by these techniques are called *exact equations*. Basically, the idea is pretty simple. Suppose you have an expression $\psi(x, y) = c$ where c is a constant. Then, ψ can describe functions $y = f(x)$ implicitly, as we noted earlier. Calculating the derivative of this function gives an ODE that can be solved by reversing this process. In these questions below you will consider the general DiffyQ's that can be solved by this method.

1. (a) As a warm-up, solve

$$y' = \frac{x^2}{y(1 + x^3)}$$

using separation of variables.
(b) Second, as another warm-up, solve the initial value problem

$$y' = \frac{3x^2 + 4x + 2}{2(y - 1)}\ , \quad y(0) = 1$$

using separation of variables.

2. (a) Now consider the equation $x^2y^3 = 64$. Viewing y as a function of x defined implicitly by this equation, find a differential equation that any such function will solve. Be sure to write your ideas out carefully using the chain rule. Could you have solved this equation by separation of variables?
(b) Now try this exact same thing as in (a) for the equation $x^2y^2 + 2xy = 64$. Could you have solved this equation by separation of variables?

3. Now try to find implicitly described solutions to the following ODE's by reversing the idea that led to the ODE's in 2. Two of the three can be solved by the method used in two, and one cannot. (These are called *exact* equations.) For the one that cannot be solved in this way, give a reason that goes beyond simply pointing out that you could solve the other two.
(a) $(2x + 3y) + (3x + 4y)y' = 0$.
(b) $y' = \frac{2x-y}{2x+y}$
(c) $\frac{x}{(x^2+y^2)^{3/2}} + \frac{y}{(x^2+y^2)^{3/2}}y' = 0$.

4. Solve

$$(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x) + (xe^{xy}\cos 2x - 3)y' = 0$$

using the above ideas. It is messy, but exact.

5. (a) Here is another approach for solving ODE's, when the equation is *homogeneous*. Consider the ODE

$$\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2}$$

expand the fraction as a sum of two fractions and make the substitution $v = y/x$. Now you need to make the appropriate change for $\frac{dy}{dx}$. If you write $y = xv$, the product rule tells you $\frac{dy}{dx} = v + x\frac{dv}{dx}$. You now obtain a differential equation you can solve between x and v. Solve it using separation of variables. You will get some logs, but stick with it. After combining the logs and taking an exponential you should be able to get rid of the logs. After replacing v by y/x again you should find the solution is

$$y = \frac{cx^2}{1 - cx}$$

where c is a constant.
(b) Use the ideas in (a) to solve $2y - xy' = 0$.

Review Problems involving Symbolic Calculation and quick slope field sketches.

Note: These Review problems will provide practice for the in-class part of the Midterm on November 15. Probably the best way to study for November 15 is to make a list of the symbolic techniques we have developed and then make up problems that can be solved by those tools. Once you do that you will see that only very special ODE can be solved with these tools and you will pretty much know what will be asked.

1. Find all solutions to the following ODEs:
(a) $y' - y = 3e^t$.
(b) $y'' + 4y = 0$.
(c) $xy' + y = x\cos x$.
(d) $(3y^2 + 2xy + x^2)y' + y^2 + 2xy + 3x^2 = 0$.

2. Draw slope fields to qualitatively give an idea of what the solutions to the following ODE look like. Then find a solution symbolically.
(a) $y' = \frac{2xy}{x^2-1}$
(b) $y' + xy = 3x$.

3. Use the ideas in problem 5 above, solve the following homogeneous equations:
(a) $y' = \frac{x+y}{x}$.
(b) $y' = -\frac{4x-3y}{2x-y}$.