Math 3CI More about solving DiffyQ III

Here you are pushed to develop some symbolic techniques for solving ODE's that go beyond the Chain-rule based techniques that gave implicit descriptions of the functions. There is actually quite a bit here and I suggest that you work on them outside of class in sequence with a classmate. Remember at all times, you have to justify approaches, not just push symbols around and integrate here and there. But at the same time, you must play with the equations (see hints) so don't be shy to try things and then work on justifications when you seem to have it.

1. Linear ODE's are equations that look like this:

$$\begin{aligned} y' + p_0(t)y &= f(t) \\ y'' + p_1(t)y' + p_0(t)y &= f(t) \\ y''' + p_2(t)y'' + p_1(t)y' + p_0(t)y &= f(t) \\ \text{ETC.} \end{aligned}$$

Of course they can have even more derivatives. These examples have what we call order *one*, *two*, *three*. Note that the functions $p_i(t)$ and $f(t)$ are functions of only the variable t. Note that there are no squares or other funny things involving y. That part of why these equations are called *linear*. One other thing while we are making definitions, we say the equation is *homogeneous* when the function $f(t) = 0$.

In an earlier homework you found some solutions to some linear ODE's by guess and check. In this assignement you will discover much more about them.

(a) Based on the above examples, write out a definition of what a general linear ODE is.

(b) Show the following: If y_1 and if y_2 are both soltuions to the same homogeneous linear ODE, and if $r \in \mathbf{R}$ is a real number, then $y_1 + y_2$ and ry_1 are also solutions to this homogeneous ODE.

(c) Show the following: If y_1 is a solution to the linear ODE $y'' + p_1(t)y' + p_0(t)y = f(t)$ and if y_2 is a soltuion to the associated homogeneous ODE $y'' + p_1(t)y' + p_0(t)y = 0$, then $y_1 + y_2$ is a solution to the first ODE.

2. The ODE $y' + p_0(t)y = 0$ can be solved by our chain rule method and by integrating both y'/y and p_0. Write a general formula for this calculation,

and since you don't know what $p_0(t)$ is you can have an $\int$ sign in your answer. Now, to see you can apply this formula, find a solution to $y' + 2ty = 0$.

3. Louis Lagrange (1736-1818) found a solution to the general equation $y' + p(t)y = f(t)$. His solution was

$$y = e^{-\int p(t)dt} \int f(t)e^{\int p(t)dt}dt$$

It looks messy but you should actually be able to check it by carefully thinking about what it means. Try to check why it works. We will work as a class on how to derive it. If you get stuck don't spend a lot of time out of class on it. Go on to the next two problems.

4. Here is another method to find a solution to a linear equation $y' + p(t)y = f(t)$. It is called finding an *integrating factor*. For this, assume you can calculate an antiderivative $\int p(t)dt$. Now, mutliply the ODE by the integrating factor $e^{\int p(t)dt}$ to obtain

$$e^{\int p(t)dt}(y' + p(t)y) = (e^{\int p(t)dt})f(y)$$

The left hand side is the derivative of $e^{\int p(t)dt}y$ so you can solve the original ODE by another integration. Now this is a lot of stuff, initially, so first try this in the following examples (the integrating factor is given for the first one).
(a) Find a solution to $y' - y = t$ using the integrating factor e^{-t}. (You can also find a solution by guess and check, but try to use the integrating factor.)
(b) Use an integrating factor to show that $y = e^{-t}\ln(1 + e^t)$ is a solution to $y' + y = 1/(1 + e^t)$.

5. Find a solution to $y' + 2ty = t^3$ where the function y satisfies the condition that $y(1) = 1$. You will need to make use of the techniques in 4 (or perhaps guess and check) *and* you will have to use the big ideas in problems 1(c) and 2 to make it work.

Extra Problem. Recall the logistic equation for rabbit population growth with limited food supply: $P'(t) = (0.1)[(100,000 - P(t))/(100,000)]P(t)$. We approximated the solution to this problem using a slope field. It is possible to solve this equation with an explicit formula if you separate variables, use partial fractions, and integrate. Try to find a formula that gives a solution to this equation. (If you like, solve the general case $y' = r(1 - y/L)y$ where r and L are real constants. Then substitute back the numbers in the Rabbit case). We predicted that it would take five to six years for the population to reach $90,000$ when $P(0) = 2,000$. How close were we to the "continuous" solution?