ASSIGNMENT 6 due date: Friday, October 12th

DETERMINANTS

With each square matrix it is possible to associate a real number called the determinant of the matrix. The value of this number will tell us whether or not the matrix is singular.

In Section 1 the definition of the determinant of a matrix is given. In Section 2 we study properties of determinants and derive an elimination method for evaluating determinants. The elimination method is generally the simplest method to use for evaluating the determinant of an $n \times n$ matrix when $n > 3$. In Section 3 we see how determinants can be applied to solving $n \times n$ linear systems and how they can be used to calculate the inverse of a matrix. An application involving cryptography is also presented in Section 3. Further applications of determinants are presented in Chapters 3 and 6.

THE DETERMINANT OF A MATRIX

With each $n \times n$ matrix A it is possible to associate a scalar, $\det(A)$, whose value will tell us whether or not the matrix is nonsingular. Before proceeding

Case 1. 1×1 Matrices

If $A = (a)$ is a 1×1 matrix, then A will have a multiplicative inverse if and only if $a \neq 0$. Thus, if we define

$$\det(A) = a$$

then A will be nonsingular if and only if $\det(A) \neq 0$.

Case 2. 2×2 Matrices

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

By Theorem 1.4.3, A will be nonsingular if and only if it is row equivalent to I. Then if $a_{11} \neq 0$, we can test whether or not A is row equivalent to I by performing the following operations:

1. Multiply the second row of A by a_{11}

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{11}a_{21} & a_{11}a_{22} \end{pmatrix}$$

2. Subtract a_{21} times the first row from the new second row

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{11}a_{22} - a_{21}a_{12} \end{pmatrix}$$

Since $a_{11} \neq 0$, the resulting matrix will be row equivalent to I if and only if

$$a_{11}a_{22} - a_{21}a_{12} \neq 0 \qquad (1)$$

If $a_{11} = 0$, we can switch the two rows of A. The resulting matrix

$$\begin{pmatrix} a_{21} & a_{22} \\ 0 & a_{12} \end{pmatrix}$$

will be row equivalent to I if and only if $a_{21}a_{12} \neq 0$. This requirement is equivalent to condition (1) when $a_{11} = 0$. Thus if A is any 2×2 matrix and we define

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

then A is nonsingular if and only if $\det(A) \neq 0$.

Notation. One can refer to the determinant of a specific matrix by enclosing the array between vertical lines. For example, if

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$$

then

$$\begin{vmatrix} 3 & 4 \end{vmatrix}$$

represents the determinant of A.

Case 3. 3×3 Matrices

We can test whether or not a 3×3 matrix is nonsingular by performing row operations to see if the matrix is row equivalent to the identity matrix I. To carry out the elimination in the first column of an arbitrary 3×3 matrix A let us first assume $a_{11} \neq 0$. The elimination can then be performed by subtracting a_{21}/a_{11} times the first row from the second and a_{31}/a_{11} times the first row from the third.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \dfrac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \dfrac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \\ 0 & \dfrac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}} & \dfrac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} \end{pmatrix}$$

The matrix on the right will be row equivalent to I if and only if

$$a_{11} \begin{vmatrix} \dfrac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \dfrac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \\ \dfrac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}} & \dfrac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} \end{vmatrix} \neq 0$$

Although the algebra is somewhat messy, this condition can be simplified to

$$\text{(2)} \qquad \begin{aligned} a_{11}a_{22}a_{33} &- a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} \\ &+ a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \neq 0 \end{aligned}$$

Thus if we define

$$\text{(3)} \qquad \begin{aligned} \det(A) = {} & a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} \\ & + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \end{aligned}$$

then for the case $a_{11} \neq 0$ the matrix will be nonsingular if and only if $\det(A) \neq 0$.

What if $a_{11} = 0$? Consider the following possibilities:

(i) $a_{11} = 0, a_{21} \neq 0$
(ii) $a_{11} = a_{21} = 0, a_{31} \neq 0$
(iii) $a_{11} = a_{21} = a_{31} = 0$

In case (i), it is not difficult to show that A is row equivalent to I if and only if

$$-a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \neq 0$$

But this condition is the same as condition (2) with $a_{11} = 0$. The details of

In case (ii) it follows that

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is row equivalent to I if and only if

$$a_{31}(a_{12}a_{23} - a_{22}a_{13}) \neq 0$$

Again this is a special case of condition (2) with $a_{11} = a_{21} = 0$.

Clearly, in case (iii) the matrix A cannot be row equivalent to I and hence must be singular. In this case if one sets a_{11}, a_{21}, and a_{31} equal to 0 in formula (3), the result will be $\det(A) = 0$.

In general, then, formula (2) gives a necessary and sufficient condition for a 3×3 matrix A to be nonsingular (regardless of the value of a_{11}).

We would now like to define the determinant of an $n \times n$ matrix. To see how to do this, note that the determinant of a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

can be defined in terms of the two 1×1 matrices

$$M_{11} = (a_{22}) \qquad \text{and} \qquad M_{12} = (a_{21})$$

The matrix M_{11} is formed from A by deleting its first row and first column and M_{12} is formed from A by deleting its first row and second column.

The determinant of A can be expressed in the form

$$\text{(4)} \qquad \det(A) = a_{11}a_{22} - a_{12}a_{21} = a_{11}\det(M_{11}) - a_{12}\det(M_{12})$$

For a 3×3 matrix A we can rewrite equation (3) in the form

$$\det(A) = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

For $j = 1, 2, 3$ let M_{1j} denote the 2×2 matrix formed from A by deleting its first row and jth column. The determinant of A can then be represented in the form

$$\text{(5)} \qquad \det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + a_{13}\det(M_{13})$$

where

$$M_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}, \quad M_{12} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}, \quad M_{13} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

To see how to generalize (4) and (5) to the case $n > 3$, we introduce the following definition.

Definition. Let $A = (a_{ij})$ be an $n \times n$ matrix. Let M_{ij} be the $(n-1) \times (n-1)$

determinant of M_{ij} is called the **minor** of a_{ij}. We define the **cofactor** A_{ij} of a_{ij} by

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

In view of this definition, for a 2×2 matrix A, we may rewrite equation (4) in the form

$$\text{(6)} \qquad \det(A) = a_{11}A_{11} + a_{12}A_{12} \qquad (n = 2)$$

Equation (6) is called the *cofactor expansion* of $\det(A)$ along the first row of A. Note that we could also write

$$\text{(7)} \qquad \det(A) = a_{21}(-a_{12}) + a_{22}a_{11} = a_{21}A_{21} + a_{22}A_{22}$$

Equation (7) expresses $\det(A)$ in terms of the entries of the second row of A and their cofactors. Actually, there is no reason why we must expand along a row of the matrix; the determinant could just as well be represented by the cofactor expansion along one of the columns.

$$\begin{aligned} \det(A) &= a_{11}a_{22} + a_{21}(-a_{12}) \\ &= a_{11}A_{11} + a_{21}A_{21} \qquad \text{(first column)} \\ \det(A) &= a_{12}(-a_{21}) + a_{22}a_{11} \\ &= a_{12}A_{12} + a_{22}A_{22} \qquad \text{(second column)} \end{aligned}$$

For a 3×3 matrix A, we have

$$\text{(8)} \qquad \det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Thus the determinant of a 3×3 matrix can be defined in terms of the elements in the first row of the matrix and their corresponding cofactors.

EXAMPLE 1. If

$$A = \begin{pmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{pmatrix}$$

then

$$\begin{aligned} \det(A) &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= (-1)^2 a_{11} \det(M_{11}) + (-1)^3 a_{12} \det(M_{12}) \\ &\quad + (-1)^4 a_{13} \det(M_{13}) \\ &= 2\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} - 5\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 4\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} \\ &= 2(6 - 8) - 5(18 - 10) + 4(12 - 5) \end{aligned}$$

As in the case of 2×2 matrices, the determinant of a 3×3 matrix can be represented as a cofactor expansion using any row or column. For example, equation (3) can be rewritten in the form

$$\begin{aligned}
\det(A) &= a_{12}a_{31}a_{23} - a_{13}a_{31}a_{22} - a_{11}a_{32}a_{23} + a_{13}a_{21}a_{32} + a_{11}a_{22}a_{33} \\
&\quad - a_{12}a_{21}a_{33} \\
&= a_{31}(a_{12}a_{23} - a_{31}a_{22}) - a_{32}(a_{11}a_{23} - a_{13}a_{21}) \\
&\quad + a_{33}(a_{11}a_{22} - a_{12}a_{21}) \\
&= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}
\end{aligned}$$

This is the cofactor expansion along the third row of A.

EXAMPLE 2. Let A be the matrix in Example 1. The cofactor expansion of $\det(A)$ along the second column is given by

$$\begin{aligned}
\det(A) &= -5\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} \\
&= -5(18-10) + 1(12-20) - 4(4-12) \\
&= -16
\end{aligned}$$

□

The determinant of a 4×4 matrix can be defined in terms of a cofactor expansion along any row or column. To compute the value of the 4×4 determinant, one would have to evaluate four 3×3 determinants.

Definition. The **determinant** of an $n \times n$ matrix A, denoted $\det(A)$, is a scalar associated with the matrix A that is defined inductively as follows:

$$\det(A) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \text{if } n > 1 \end{cases}$$

where

$$A_{1j} = (-1)^{1+j}\det(M_{1j}) \qquad j = 1, \ldots, n$$

are the cofactors associated with the entries in the first row of A.

As we have seen, it is not necessary to limit ourselves to using the first row for the cofactor expansion. We state the following theorem without proof.

Theorem 2.1.1. *If A is an $n \times n$ matrix with $n \geq 2$, then* $\det(A)$ *can be expressed as a cofactor expansion using any row or column of A.*

$$\begin{aligned}
\det(A) &= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \\
&= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}
\end{aligned}$$

The cofactor expansion of a 4×4 determinant will involve four 3×3 determinants. One can often save work by expanding along the row or column that contains the most zeros. For example, to evaluate

$$\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{vmatrix}$$

one would expand down the first column. The first three terms will drop out, leaving

$$-2\begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix} = -2 \cdot 3 \cdot \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 12$$

The cofactor expansion can be used to establish some important results about determinants. These results are given in the following theorems.

Theorem 2.1.2. *If A is an $n \times n$ matrix, then* $\det(A^T) = \det(A)$.

Proof. The proof is by induction on n. Clearly, the result holds if $n = 1$, since a 1×1 matrix is necessarily symmetric. Assume that the result holds for all $k \times k$ matrices and that A is a $(k+1) \times (k+1)$ matrix. Expanding $\det(A)$ along the first row of A, we get

$$\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + - \cdots \pm a_{1,k+1}\det(M_{1,k+1})$$

Since the M_{ij}'s are all $k \times k$ matrices, it follows from the induction hypothesis that

(9) $\det(A) = a_{11}\det(M_{11}^T) - a_{12}\det(M_{12}^T) + - \cdots \pm a_{1,k+1}\det(M_{1,k+1}^T)$

The right-hand side of (9) is just the expansion by minors of $\det(A^T)$ using the first column of A^T. Therefore,

$$\det(A^T) = \det(A)$$ □

Theorem 2.1.3. *If A is an $n \times n$ triangular matrix, the determinant of A equals the product of the diagonal elements of A.*

Proof. In view of Theorem 2.1.2, it suffices to prove the theorem for lower triangular matrices. The result follows easily using the cofactor expansion and induction on n. The details of this are left for the reader (see Exercise 8). □

Theorem 2.1.4. *Let A be an $n \times n$ matrix.*

(i) *If A has a row or column consisting entirely of zeros, then* $\det(A) = 0$.

(ii) *If A has two identical rows or two identical columns, then* $\det(A) = 0$.

PROPERTIES OF DETERMINANTS

In this section we consider the effects of row operations on the determinant of a matrix. Once these effects have been established, we will prove that a matrix A is singular if and only if its determinant is zero and we will develop a method for evaluating determinants using row operations. Also, we will establish an important theorem about the determinant of the product of two matrices. We begin with the following lemma.

Lemma 2.2.1. *Let A be an $n \times n$ matrix. If A_{jk} denotes the cofactor of a_{jk} for $k = 1, \ldots, n$, then*

$$(1) \qquad a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Proof. If $i = j$, (1) is just the cofactor expansion of $\det(A)$ along the ith row of A. To prove (1) in the case $i \neq j$, let A^* be the matrix obtained by replacing the jth row of A by the ith row of A.

$$A^* = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad j\text{th row}$$

Since two rows of A^* are the same, its determinant must be zero. It follows from the cofactor expansion of $\det(A^*)$ along the jth row that

$$\begin{aligned} 0 = \det(A^*) &= a_{i1}A^*_{j1} + a_{i2}A^*_{j2} + \cdots + a_{in}A^*_{jn} \\ &= a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} \end{aligned}$$ □

Let us now consider the effects of each of the three row operations on the value of the determinant. We start with row operation II.

ROW OPERATION II

A row of A is multiplied by a nonzero constant.

Let E denote the elementary matrix of type II formed from I by multiplying along the ith row, then

$$\begin{aligned}\det(EA) &= \alpha a_{i1}A_{i1} + \alpha a_{i2}A_{i2} + \cdots + \alpha a_{in}A_{in} \\ &= \alpha(a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}) \\ &= \alpha \det(A)\end{aligned}$$

In particular,

$$\det(E) = \det(EI) = \alpha \det(I) = \alpha$$

and hence

$$\det(EA) = \alpha \det(A) = \det(E)\det(A)$$

ROW OPERATION III

A multiple of one row is added to another row.

Let E be the elementary matrix of type III formed from I by adding c times the ith row to the jth row. Since E is triangular and its diagonal elements are all 1, it follows that $\det(E) = 1$. We will show that

$$\det(EA) = \det(A) = \det(E)\det(A)$$

If $\det(EA)$ is expanded by cofactors along the jth row, it follows from Lemma 2.2.1 that

$$\begin{aligned}\det(EA) &= (a_{j1} + ca_{i1})A_{j1} + (a_{j2} + ca_{i2})A_{j2} \\ &\quad + \cdots + (a_{jn} + ca_{in})A_{jn} \\ &= (a_{j1}A_{j1} + \cdots + a_{jn}A_{jn}) \\ &\quad + c(a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}) \\ &= \det(A)\end{aligned}$$

Thus

$$\det(EA) = \det(A) = \det(E)\det(A)$$

ROW OPERATION I

Two rows of A are interchanged.

To see the effects of row operation I, we note that this operation can be accomplished using row operations II and III. We illustrate how this is done for 3×3 matrices.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \end{pmatrix}$$

Subtracting row 3 from row 2 yields

$$A^{(1)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - a_{31} & a_{22} - a_{32} & a_{23} - a_{33} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Next, the second row of $A^{(1)}$ is added to the third row:

$$A^{(2)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - a_{31} & a_{22} - a_{32} & a_{23} - a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Subtracting row 3 from row 2, we get

$$A^{(3)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -a_{31} & -a_{32} & -a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Since all of these matrices have been formed using only row operation III, it follows that

$$\det(A) = \det(A^{(1)}) = \det(A^{(2)}) = \det(A^{(3)})$$

Finally, if the second row of A is multiplied through by -1, one obtains

$$A^{(4)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Since row operation II was used, it follows that

$$\det(A^{(4)}) = -1 \det(A^{(3)}) = -\det(A)$$

$A^{(4)}$ is just the matrix obtained by interchanging the second and third rows of A.

This same argument can be applied to $n \times n$ matrices to show that whenever two rows are switched the sign of the determinant is changed. Thus if A is $n \times n$ and E_{ij} is the $n \times n$ elementary matrix formed by interchanging the ith and jth rows of I, then

$$\det(E_{ij}A) = -\det(A)$$

In particular,

$$\det(E_{ij}) = \det(E_{ij}I) = -\det(I) = -1$$

Thus for any elementary matrix E of type I,

$$\det(EA) = -\det(A) = \det(E)\det(A)$$

In summation, if E is an elementary matrix, then

$$\det(EA) = \det(E)\det(A)$$

where

$$(2) \qquad \det(E) = \begin{cases} -1 & \text{if } E \text{ is of type I} \\ \alpha \neq 0 & \text{if } E \text{ is of type II} \\ 1 & \text{if } E \text{ is of type III} \end{cases}$$

Similar results hold for column operations. Indeed, if E is an elementary matrix, then

$$\begin{aligned} \det(AE) &= \det\big((AE)^T\big) = \det\big(E^T A^T\big) \\ &= \det\big(E^T\big)\det\big(A^T\big) = \det(E)\det(A) \end{aligned}$$

Thus the effects that row or column operations have on the value of the determinant can be summarized as follows:

I. Interchanging two rows (or columns) of a matrix changes the sign of the determinant.
II. Multiplying a single row or column of a matrix by a scalar has the effect of multiplying the value of the determinant by that scalar.
III. Adding a multiple of one row (or column) to another does not change the value of the determinant.

Note. As a consequence of III, if one row (or column) of a matrix is a multiple of another, the determinant of the matrix must equal zero.

If follows from (2) that all elementary matrices have nonzero determinants. This observation can be used to prove the following theorem.

Theorem 2.2.2. *An $n \times n$ matrix A is singular if and only if*

$$\det(A) = 0$$

Proof. The matrix A can be reduced to row echelon form with a finite number of row operations. Thus

$$U = E_k E_{k-1} \cdots E_1 A$$

where U is in row echelon form and the E_i's are all elementary matrices.

$$\begin{aligned} \det(U) &= \det(E_k E_{k-1} \cdots E_1 A) \\ &= \det(E_k)\det(E_{k-1}) \cdots \det(E_1)\det(A) \end{aligned}$$

Since the determinants of the E_i's are all nonzero, it follows that $\det(A) = 0$ if and only if $\det(U) = 0$. If A is singular, then U has a row consisting entirely of zeros and hence $\det(U) = 0$. If A is nonsingular, U is triangular with 1's along the diagonal and hence $\det(U) = 1$. □

From the proof of Theorem 2.2.2 we can obtain a method for computing det(A). Reduce A to row echelon form.

$$U = E_k E_{k-1} \cdots E_1 A$$

If the last row of U consists entirely of zeros, A is singular and $\det(A) = 0$. Otherwise, A is nonsingular and

$$\det(A) = [\det(E_k)\det(E_{k-1}) \cdots \det(E_1)]^{-1}$$

Actually, if A is nonsingular, it is simpler to reduce A to triangular form. This can be done using only row operations I and III. Thus

$$T = E_m E_{m-1} \cdots E_1 A$$

and hence

$$\det(A) = \pm\det(T) = \pm t_{11} t_{22} \cdots t_{nn}$$

The sign will be positive if row operation I has been used an even number of times and negative otherwise.

EXAMPLE 1. Evaluate

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix}$$

SOLUTION

$$\begin{aligned} \begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} &= \begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{vmatrix} \\ &= (-1)\begin{vmatrix} 2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5 \end{vmatrix} \\ &= (-1)(2)(-6)(-5) \\ &= -60 \end{aligned}$$

□

We now have two methods for evaluating the determinant of an $n \times n$ matrix A. If $n > 3$ and A has nonzero entries, elimination is the most efficient method in the sense that it involves less arithmetic operations. In Table 1 the number of arithmetic operations involved in each method is given for $n = 2, 3, 4, 5, 10$. It is not difficult to derive general formulas for the number of operations in each of the methods (see Exercises 16 and 17).

TABLE 1

	Cofactors		Elimination	
n	Additions	Multiplications	Additions	Multiplications and Divisions
2	1	2	1	3
3	5	9	5	10
4	23	40	14	23
5	119	205	30	45
10	3,628,799	6,235,300	285	339

We have seen that for any elementary matrix E,

$$\det(EA) = \det(E)\det(A) = \det(AE)$$

This is a special case of the following theorem.

Theorem 2.2.3. *If A and B are $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B)$$

Proof. If B is singular, it follows from Theorem 1.4.3 that AB is also singular (see Exercise 15 of Chapter 1, Section 4), and therefore

$$\det(AB) = 0 = \det(A)\det(B)$$

If B is nonsingular, B can be written as a product of elementary matrices. We have already seen that the result holds for elementary matrices. Thus

$$\begin{aligned}\det(AB) &= \det(AE_kE_{k-1}\cdots E_1)\\ &= \det(A)\det(E_k)\det(E_{k-1})\cdots\det(E_1)\\ &= \det(A)\det(E_kE_{k-1}\cdots E_1)\\ &= \det(A)\det(B)\end{aligned}$$

□

If A is singular, the computed value of $\det(A)$ using exact arithmetic must be 0. However, this result is unlikely if the computations are done by computer. Since computers use a finite number system, roundoff errors are usually unavoidable. Consequently, it is more likely that the computed value of $\det(A)$ will only be near 0. Because of roundoff errors, it is virtually impossible to determine computationally whether or not a matrix is exactly singular. In computer applications it is often more meaningful to ask whether a matrix is "close" to being singular. In general, the value of $\det(A)$ is not a good indicator of nearness to singularity. In Chapter 7 we will discuss how to determine whether or not a matrix is close to being singular.

EXERCISES

1. Evaluate each of the following determinants by inspection.

(a) $\begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

2. Let

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{pmatrix}$$

(a) Use the elimination method to evaluate $\det(A)$.

(b) Use the value of $\det(A)$ to evaluate

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 4 & 4 \\ 2 & 3 & -1 & -2 \end{vmatrix}$$

3. For each of the following, compute the determinant and state whether the matrix is singular or nonsingular.

(a) $\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 5 \\ 2 & 1 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{pmatrix}$

4. Find all possible choices of c that would make the following matrix singular.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{pmatrix}$$

5. Let A be an $n \times n$ matrix and α a scalar. Show that

$$\det(\alpha A) = \alpha^n \det(A)$$

6. Let A be a nonsingular matrix. Show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

7. Let A and B be 3×3 matrices with $\det(A) = 4$ and $\det(B) = 5$. Find the value of:
(a) $\det(AB)$ (b) $\det(3A)$ (c) $\det(2AB)$ (d) $\det(A^{-1}B)$

8. Let E_1, E_2, E_3 be 3×3 elementary matrices of types I, II, and III, respectively, and let A be a 3×3 matrix with $\det(A) = 6$. Assume, additionally, that E_2 was formed from I by multiplying its second row by 3. Find the values of each of the following.
(a) $\det(E_1 A)$ (b) $\det(E_2 A)$ (c) $\det(E_3 A)$
(d) $\det(A E_1)$ (e) $\det(E_1^2)$ (f) $\det(E_1 E_2 E_3)$

9. Let A and B be row equivalent matrices and suppose that B can be obtained from A using only row operations I and III. How do the values of $\det(A)$ and $\det(B)$ compare? How will the values compare if B can be obtained from A using only row operation III? Explain your answers.

10. Consider the 3×3 Vandermonde matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$$

(a) Show that $\det(V) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$.
(b) What conditions must the scalars x_1, x_2, x_3 satisfy in order for V to be nonsingular?

11. Suppose that a 3×3 matrix A factors into a product

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$