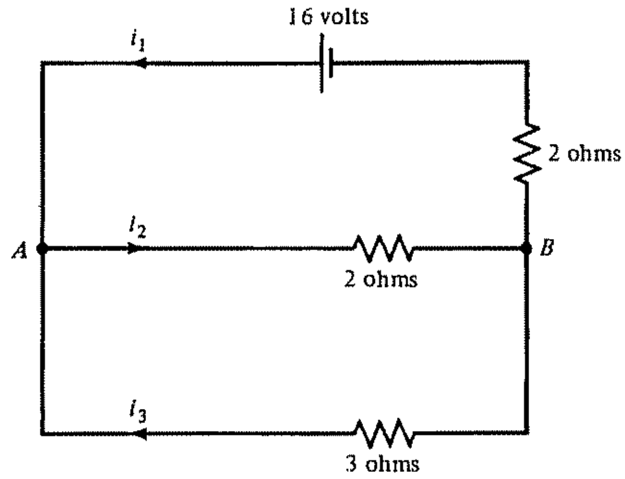
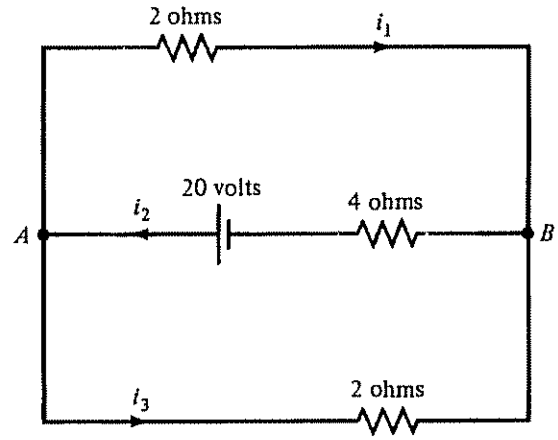
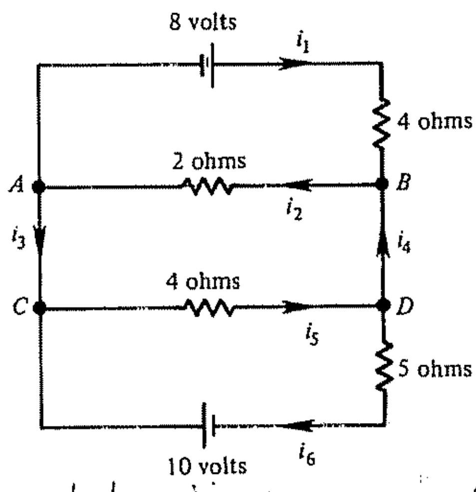
(a)



(b)



(c)



ASSIGNMENT 5 due date: Wednesday, Oct. 10

MATRIX ALGEBRA

In this section we define arithmetic operations with matrices and look at some of their algebraic properties. Matrices are one of the most powerful tools in mathematics. To use matrices effectively, we must be adept at matrix arithmetic.

The entries of a matrix are called *scalars*. They are usually either real or complex numbers. For the most part we will be working with matrices whose entries are real numbers. Throughout the first five chapters of the book the reader may assume that the term *scalar* refers to a real number. However, in Chapter 6 there will be occasions when we will use the set of complex numbers as our scalar field.

If we wish to refer to matrices without specifically writing out all their entries, we will use capital letters A, B, C, and so on. In general, a_{ij} will denote the entry of the matrix A that is in the ith row and the jth column. Thus if A is an $m \times n$ matrix, then

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

We will sometimes shorten this to $A = (a_{ij})$. Similarly, a matrix B may be referred to as (b_{ij}), a matrix C as (c_{ij}), and so on.

EQUALITY

Definition. Two $m \times n$ matrices A and B are said to be **equal** if $a_{ij} = b_{ij}$ for each i and j.

SCALAR MULTIPLICATION

If A is a matrix and α is a scalar, then αA is the matrix formed by multiplying each of the entries of A by α. For example, if

$$A = \begin{pmatrix} 4 & 8 & 2 \\ 6 & 8 & 10 \end{pmatrix}$$

then

$$\tfrac{1}{2}A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 4 & 5 \end{pmatrix} \qquad \text{and} \qquad 3A = \begin{pmatrix} 12 & 24 & 6 \\ 18 & 24 & 30 \end{pmatrix}$$

MATRIX ADDITION

If $A = (a_{ij})$ and $B = (b_{ij})$ are both $m \times n$ matrices, then the sum $A + B$ is the $m \times n$ matrix whose ijth entry is $a_{ij} + b_{ij}$ for each ordered pair (i, j). For example,

$$\begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 3 \\ 5 & 7 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix} + \begin{pmatrix} -8 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ 10 \end{pmatrix}$$

If we define $A - B$ to be $A + (-1)B$, then it turns out that $A - B$ is formed by subtracting the corresponding entry of B from each entry of A. Thus

$$\begin{aligned}\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix} &= \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} + (-1)\begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} -4 & -5 \\ -2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 2-4 & 4-5 \\ 3-2 & 1-3 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}\end{aligned}$$

If O represents a matrix, with the same dimensions as A, whose entries are all 0, then

$$A + O = O + A = A$$

That is, the zero matrix acts as an additive identity on the set of all $m \times n$ matrices. Furthermore, each $m \times n$ matrix A has an additive inverse. Indeed,

$$A + (-1)A = O = (-1)A + A$$

It is customary to denote the additive inverse by $-A$. Thus

$$-A = (-1)A$$

MATRIX MULTIPLICATION

We have yet to define the most important operation, the multiplication of two matrices. Much of the motivation behind the definition comes from the applications to linear systems of equations. If we have a system of one linear equation in one unknown, it can be written in the form

(1) $$ax = b$$

We generally think of a, x, and b as being scalars; however, they could also be treated as 1×1 matrices. More generally, given an $m \times n$ linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

it is desirable to write the system in a form similar to (1), that is, as a matrix equation

$$AX = B$$

where $A = (a_{ij})$ is known, X is an $n \times 1$ matrix of unknowns, and B is an $m \times 1$ matrix representing the right-hand side of the system. Thus we set

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

and

$$(2) \qquad AX = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

Given an $m \times n$ matrix A and an $n \times 1$ matrix X it is possible to compute a product AX by (2). The product AX will be an $m \times 1$ matrix. The rule for determining the ith entry of AX is

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

Note that the ith entry is determined using only the ith row of A. The entries in that row are paired off with the corresponding entries of X and multiplied. The n products are then summed. Those readers familiar with dot products will recognize this as simply the dot product of the n-tuple corresponding to the ith row of A with the n-tuple corresponding to the matrix X.

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

In order to pair off the entries in this way, the number of columns of A must equal the number of rows of X. The entries of X can be either scalars or unknowns having scalar values.

EXAMPLE 1

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 5 & 3 & 7 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$AX = \begin{pmatrix} 4x_1 + 2x_2 + x_3 \\ 5x_1 + 3x_2 + 7x_3 \end{pmatrix}$$

□

EXAMPLE 2

$$A = \begin{pmatrix} -3 & 1 \\ 2 & 5 \\ 4 & 2 \end{pmatrix}, \qquad X = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$AX = \begin{pmatrix} -3 \cdot 2 + 1 \cdot 4 \\ 2 \cdot 2 + 5 \cdot 4 \\ 4 \cdot 2 + 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 24 \\ 16 \end{pmatrix}$$

□

EXAMPLE 3. Write the following system of equations as a matrix equation $AX = B$.

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 5 \\ x_1 - 2x_2 + 5x_3 &= -2 \\ 2x_1 + x_2 - 3x_3 &= 1 \end{aligned}$$

SOLUTION

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & -2 & 5 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

□

More generally, it is possible to multiply a matrix A times a matrix B if the number of columns of A equals the number of rows of B. The first column of the product is determined by the first column of B, the second column by the second column of B, and so on. Thus, to determine the (i, j) entry of the product AB, we use the entries of the ith row of A and the jth column of B.

Definition. If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times r$ matrix, then the product $AB = C = (c_{ij})$ is the $m \times r$ matrix whose entries are defined by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

What this definition says is that to find the ijth element of the product, you take the ith row of A and the jth column of B, multiply the corresponding elements pairwise, and add the resulting numbers.

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}$$

NOTATIONAL RULES

Just as in ordinary algebra, if an expression involves both multiplication and addition and there are no parentheses to indicate the order of the operations, multiplications are carried out before additions. This is true for both scalar and matrix multiplications. For example, if

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix}$$

then

$$A + BC = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 7 & 7 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 11 \\ 0 & 6 \end{pmatrix}$$

and

$$3A + B = \begin{pmatrix} 9 & 12 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 15 \\ 5 & 7 \end{pmatrix}$$

ALGEBRAIC RULES

The following theorem provides some useful rules for doing matrix arithmetic.

Theorem 1.3.1. *Each of the following statements is valid for any scalars α and β and for any matrices A, B, and C for which the indicated operations are defined.*

(1) $A + B = B + A$
(2) $(A + B) + C = A + (B + C)$
(3) $(AB)C = A(BC)$
(4) $A(B + C) = AB + AC$
(5) $(A + B)C = AC + BC$
(6) $(\alpha\beta)A = \alpha(\beta A)$
(7) $\alpha(AB) = (\alpha A)B = A(\alpha B)$
(8) $(\alpha + \beta)A = \alpha A + \beta A$
(9) $\alpha(A + B) = \alpha A + \alpha B$

We will prove two of the rules and leave the rest for the reader to verify.

Proof of (4) Assume that $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ and $C = (c_{ij})$ are both $n \times r$ matrices. Let $D = A(B + C)$ and $E = AB + AC$. It follows that

$$d_{ij} = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$$

and

$$e_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$

But

$$\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$

so that $d_{ij} = e_{ij}$ and hence $A(B + C) = AB + AC$. □

Proof of (3). Let A be an $m \times n$ matrix, B an $n \times r$ matrix, and C an $r \times s$ matrix. Let $D = AB$ and $E = BC$. We must show that $DC = AE$. By the definition of matrix multiplication,

$$d_{il} = \sum_{k=1}^{n} a_{ik}b_{kl} \qquad \text{and} \qquad e_{kj} = \sum_{l=1}^{r} b_{kl}c_{lj}$$

The ijth term of DC is

$$\sum_{l=1}^{r} d_{il}c_{lj} = \sum_{l=1}^{r} \left(\sum_{k=1}^{n} a_{ik}b_{kl} \right) c_{lj}$$

and the ijth entry of AE is

$$\sum_{k=1}^{n} a_{ik}e_{kj} = \sum_{k=1}^{n} a_{ik} \left(\sum_{l=1}^{r} b_{kl}c_{lj} \right)$$

Since

$$\sum_{l=1}^{r} \left(\sum_{k=1}^{n} a_{ik}b_{kl} \right) c_{lj} = \sum_{l=1}^{r} \sum_{k=1}^{n} a_{ik}b_{kl}c_{lj} = \sum_{k=1}^{n} a_{ik} \left(\sum_{l=1}^{r} b_{kl}c_{lj} \right)$$

it follows that

$$(AB)C = DC = AE = A(BC)$$ □

The arithmetic rules given in Theorem 1.3.1 seem quite natural since they are similar to the rules we use with real numbers. However, there are some important differences between the rules for matrix arithmetic and those for real number arithmetic. In particular, multiplication of real numbers is commutative; however, we saw in Example 6 that matrix multiplication is not commutative. This difference warrants special emphasis.

> **Warning:** In general, $AB \neq BA$.
> Matrix multiplication is *not* commutative.

Some of the other differences between matrix arithmetic and real number arithmetic are illustrated in Exercises 13, 14, and 15.

EXAMPLE 7. If

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}, \qquad \text{and} \qquad C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

verify that $A(BC) = (AB)C$ and $A(B + C) = AB + AC$.

SOLUTION

$$A(BC) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 16 & 11 \end{pmatrix}$$

$$(AB)C = \begin{pmatrix} -4 & 5 \\ -6 & 11 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 16 & 11 \end{pmatrix}$$

Thus

$$A(BC) = \begin{pmatrix} 6 & 5 \\ 16 & 11 \end{pmatrix} = (AB)C$$

$$A(B + C) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 5 & 15 \end{pmatrix}$$

$$AB + AC = \begin{pmatrix} -4 & 5 \\ -6 & 11 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ 11 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 5 & 15 \end{pmatrix}$$

Therefore,

$$A(B + C) = AB + AC$$

□

Notation. Since $(AB)C = A(BC)$, one may simply omit the parentheses and write ABC. The same is true for a product of four or more matrices. In the case where an $n \times n$ matrix is multiplied by itself a number of times, it is convenient to use exponential notation. Thus if k is a positive integer, then

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}}$$

EXAMPLE 8. If

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

then

$$A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$A^3 = AAA = AA^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

and in general

$$A^n = \begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}$$

□

(a) Determine the adjacency matrix A of the graph.

(b) Compute A^2. What do the entries in the first row of A tell you about walks of length 2 that start from V_1?

(c) Compute A^3. How many walks of length 3 are there from V_2 to V_4? How many walks of length less than or equal to 3 are there from V_2 to V_4?

23. Let A be a 2×2 matrix with $a_{11} \neq 0$ and let $\alpha = a_{21}/a_{11}$. Show that A can be factored into a product of the form

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & b \end{pmatrix}$$

What is the value of b?

SPECIAL TYPES OF MATRICES

In this section we look at special types of $n \times n$ matrices, such as triangular matrices, diagonal matrices, and elementary matrices. These special types of matrices play an important role in the solution of matrix equations. We begin by considering a special matrix I that acts like a multiplicative identity, that is,

$$IA = AI = A$$

for any $n \times n$ matrix A. We also discuss the existence and computation of multiplicative inverses.

THE IDENTITY MATRIX

One very important matrix is the $n \times n$ matrix I with 1's on the diagonal and 0's off the diagonal. Thus $I = (\delta_{ij})$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

If A is any $n \times n$ matrix, $AI = IA = A$. The matrix I acts as an identity for the multiplication of $n \times n$ matrices and consequently is referred to as the *identity matrix*. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{pmatrix}$$

and

$$\begin{pmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{pmatrix}$$

In general, if B is any $m \times n$ matrix and C is any $n \times r$ matrix, then

$$BI = B \quad \text{and} \quad IC = C$$

Notation. The set of all n-tuples of real numbers is called *Euclidean n-space* and is usually denoted by R^n. The elements of R^n are called *vectors*. Note, however, that the solution to the matrix equation $AX = B$ will be an $n \times 1$ matrix rather than an n-tuple. In general, when working with matrix equations it is more convenient to think of R^n as consisting of column vectors ($n \times 1$ matrices) rather than row vectors ($1 \times n$ matrices). The standard notation for a column vector is a boldface lowercase letter.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \mathbf{x}^T = (x_1, \ldots, x_n)$$

column vector row vector

Following this convention, we will use the notation $A\mathbf{x} = \mathbf{b}$, rather than $AX = B$, to represent a linear system of equations.

Given an $m \times n$ matrix A, it is often necessary to refer to a particular row or column. The ith row vector of A will be denoted by $\mathbf{a}(i, :)$ and the jth column vector will be denoted by $\mathbf{a}(:, j)$. In general we will be working primarily with column vectors. Consequently, we will use the shorthand notation $\mathbf{a}_j$ in place of $\mathbf{a}(:, j)$. Since references to row vectors are far less frequent, we will not use any shorthand notation for row vectors. In summation, if A is an $m \times n$ matrix, then the row vectors of A are given by

$$\mathbf{a}(i, :) = (a_{i1}, a_{i2}, \ldots, a_{in}) \qquad i = 1, \ldots, m$$

and the column vectors are given by

$$\mathbf{a}_j = \mathbf{a}(:, j) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \qquad j = 1, \ldots, n$$

Similarly, if B is an $n \times r$ matrix, then $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r)$. The only exception to this notation is in the case of the identity matrix I. The standard notation for the jth column vector of I is $\mathbf{e}_j$ rather than $\mathbf{i}_j$. Thus the $n \times n$ identity matrix can be written

$$I = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$$

DIAGONAL AND TRIANGULAR MATRICES

An $n \times n$ matrix A is said to be *upper triangular* if $a_{ij} = 0$ for $i > j$ and *lower triangular* if $a_{ij} = 0$ for $i < j$. Also, A is said to be *triangular* if it is either upper triangular or lower triangular. For example, the 3×3 matrices

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 1 & 4 & 3 \end{pmatrix}$$

are both triangular. The first is upper triangular and the second is lower triangular.

A triangular matrix may have 0's on the diagonal. However, for a linear system $A\mathbf{x} = \mathbf{b}$ to be in triangular form, the coefficient matrix A must be triangular with nonzero diagonal entries.

An $n \times n$ matrix A is *diagonal* if $a_{ij} = 0$ whenever $i \neq j$. The matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are all diagonal. A diagonal matrix is both upper triangular and lower triangular.

MATRIX INVERSION

Definition. An $n \times n$ matrix A is said to be **nonsingular** or **invertible** if there exists a matrix B such that $AB = BA = I$. The matrix B is said to be a multiplicative inverse of A.

If B and C are both multiplicative inverses of A, then

$$B = BI = B(AC) = (BA)C = IC = C$$

Thus a matrix can have at most one multiplicative inverse. We will refer to the multiplicative inverse of a nonsingular matrix A as simply the *inverse* of A and denote it by A^{-1}.

EXAMPLE 1. The matrices

$$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{pmatrix}$$

are inverses of each other, since

$$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

EXAMPLE 2. The triangular matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

are inverses, since

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

□

EXAMPLE 3. The matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

has no inverse. Indeed, if B is any 2×2 matrix, then

$$BA = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b_{11} & 0 \\ b_{21} & 0 \end{pmatrix}$$

Thus BA cannot equal I. □

Definition. An $n \times n$ matrix is said to be **singular** if it does not have a multiplicative inverse.

EQUIVALENT SYSTEMS

Given an $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$, we can obtain an equivalent system by multiplying both sides of the equation by a nonsingular $m \times m$ matrix M.

(1) $$A\mathbf{x} = \mathbf{b}$$

(2) $$MA\mathbf{x} = M\mathbf{b}$$

Clearly, any solution to (1) will also be a solution to (2). On the other hand, if $\hat{\mathbf{x}}$ is a solution to (2), then

$$M^{-1}(MA\hat{\mathbf{x}}) = M^{-1}(M\mathbf{b})$$
$$A\hat{\mathbf{x}} = \mathbf{b}$$

so the two systems are equivalent.

To obtain an equivalent system that is easier to solve, we can apply a sequence of nonsingular matrices E ... E to both sides of the equation

$A\mathbf{x} = \mathbf{b}$ to obtain a simpler system

$$U\mathbf{x} = \mathbf{c}$$

where $U = E_k \cdots E_1 A$ and $\mathbf{c} = E_k \cdots E_2 E_1 \mathbf{b}$. The new system will be equivalent to the original provided that $M = E_k \cdots E_1$ is nonsingular. However, M is the product of nonsingular matrices. The following theorem shows that any product of nonsingular matrices is nonsingular.

Theorem 1.4.1. *If A and B are nonsingular $n \times n$ matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.*

Proof

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}B = I$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I \qquad \square$$

It follows by induction that if $E_1, \ldots, E_k$ are all nonsingular, then the product $E_1 E_2 \cdots E_k$ is nonsingular and

$$(E_1 E_2 \cdots E_k)^{-1} = E_k^{-1} \cdots E_2^{-1} E_1^{-1}$$

We will show next that any of the three elementary row operations can be accomplished by multiplying A on the left by a nonsingular matrix.

ELEMENTARY MATRICES

A matrix obtained from the identity matrix I by the performance of one elementary row operation is called an *elementary* matrix.

There are three types of elementary matrices corresponding to the three types of elementary row operations.

Type I

An elementary matrix of type I is a matrix obtained by interchanging two rows of I.

EXAMPLE 4. Let

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

E_1 is an elementary matrix of type I, since it was obtained by interchanging the first two rows of I. Let A be a 3×3 matrix.

$$E_1 A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$AE_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{pmatrix}$$

Multiplying A on the left by E_1 interchanges the first and second rows of A. Right multiplication of A by E_1 is equivalent to the elementary column operation of interchanging the first and second columns. □

Type II

An elementary matrix of type II is a matrix obtained by multiplying a row of I by a nonzero constant.

EXAMPLE 5

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

is an elementary matrix of type II.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 3a_{13} \\ a_{21} & a_{22} & 3a_{23} \\ a_{31} & a_{32} & 3a_{33} \end{pmatrix}$$

Multiplication on the left by E_2 performs the elementary row operation of multiplying the third row by 3, while multiplication on the right by E_2 performs the elementary column operation of multiplying the third column by 3. □

Type III

An elementary matrix of type III is a matrix obtained from I by adding a multiple of one row to another row.

EXAMPLE 6

$$E_3 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is an elementary matrix of type III. If A is a 3×3 matrix, then

$$E_3A = \begin{pmatrix} a_{11} + 3a_{31} & a_{12} + 3a_{32} & a_{13} + 3a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$AE_3 = \begin{pmatrix} a_{11} & a_{12} & 3a_{11} + a_{13} \\ a_{21} & a_{22} & 3a_{21} + a_{23} \\ a_{31} & a_{32} & 3a_{31} + a_{33} \end{pmatrix}$$

Multiplication on the left by E_3 adds 3 times the third row to the first row. Multiplication on the right adds 3 times the first column to the third column. □

In general, suppose that E is an $n \times n$ elementary matrix. We can think of E as being obtained from I by either a row operation or a column operation. If A is an $n \times r$ matrix, premultiplying A by E has the effect of performing that same row operation on A. If B is an $m \times n$ matrix, postmultiplying B by E is equivalent to performing that same column operation on B.

Theorem 1.4.2. *If E is an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type.*

Proof. If E is the elementary matrix of type I formed from I by interchanging the ith and jth rows, then E can be transformed back into I by interchanging these same rows again. Thus $EE = I$ and hence E is its own inverse. If E is the elementary matrix of type II formed by multiplying the ith row of I by a nonzero scalar α, then E can be transformed into the identity by multiplying either its ith row or its ith column by $1/\alpha$. Thus

$$E^{-1} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & O & \\ & & 1 & & & & \\ & & & 1/\alpha & & & \\ & & & & 1 & & \\ & O & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \quad i\text{th row}$$

Finally, suppose that E is the elementary matrix of type III formed from I by adding m times the ith row to the jth row.

$$E = \begin{pmatrix} 1 & & & & & & \\ \vdots & \ddots & & & & O & \\ 0 & \cdots & 1 & & & & \\ \vdots & & & \ddots & & & \\ 0 & \cdots & m & \cdots & 1 & & \\ \vdots & & & & & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \quad \begin{matrix} \\ \\ i\text{th row} \\ \\ j\text{th row} \\ \\ \\ \end{matrix}$$

E can be transformed back into I by either subtracting m times the ith row column. Thus

$$E^{-1} = \begin{pmatrix} 1 & & & & & & \\ \vdots & \ddots & & & & O & \\ 0 & \cdots & 1 & & & & \\ \vdots & & & \ddots & & & \\ 0 & \cdots & -m & \cdots & 1 & & \\ \vdots & & & & & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}$$ □

Definition. A matrix B is **row equivalent** to A if there exists a finite sequence $E_1, E_2, \ldots, E_k$ of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_1 A$$

In other words, B is row equivalent to A if B can be obtained from A by a finite number of row operations. In particular, two augmented matrices $(A \mid \mathbf{b})$ and $(B \mid \mathbf{c})$ are row equivalent if and only if $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{c}$ are equivalent systems.

The following properties of row equivalent matrices are consequences of Theorem 1.4.2.

(i) If A is row equivalent to B, then B is row equivalent to A.

(ii) If A is row equivalent to B, and B is row equivalent to C, then A is row equivalent to C.

The details of the proofs of (i) and (ii) are left as an exercise for the reader.

Theorem 1.4.3. *Let A be an $n \times n$ matrix. The following are equivalent:*

(a) *A is nonsingular.*

(b) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$.*

(c) *A is row equivalent to I.*

Proof. We prove first that statement (a) implies statement (b). If A is nonsingular and $\hat{\mathbf{x}}$ is a solution to $A\mathbf{x} = \mathbf{0}$, then

$$\hat{\mathbf{x}} = I\hat{\mathbf{x}} = (A^{-1}A)\hat{\mathbf{x}} = A^{-1}(A\hat{\mathbf{x}}) = A^{-1}\mathbf{0} = \mathbf{0}$$

Thus $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Next we show that statement (b) implies statement (c). If we use elementary row operations, the system can be transformed into the form $U\mathbf{x} = \mathbf{0}$, where U is in row echelon form. If one of the diagonal elements of U were 0, the last row of U would consist entirely of 0's. But then $A\mathbf{x} = \mathbf{0}$ would be equivalent to a system with more unknowns than equations and hence by Theorem 1.2.1 would have a nontrivial solution. Thus U must be a triangular matrix with diagonal elements all equal to 1. It follows then that I is the reduced row echelon form of A and hence A is row

Finally, we will show that statement (c) implies statement (a). If A is row equivalent to I, there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$A = E_k E_{k-1} \cdots E_1 I = E_k E_{k-1} \cdots E_1$$

But since E_i is invertible, $i = 1, \ldots, k$, the product $E_k E_{k-1} \cdots E_1$ is also invertible. Hence A is nonsingular and

$$A^{-1} = (E_k E_{k-1} \cdots E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1} \qquad \square$$

Corollary 1.4.4. *The system of n linear equations in n unknowns $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if A is nonsingular.*

Proof. If A is nonsingular, then $A^{-1}\mathbf{b}$ is the only solution to $A\mathbf{x} = \mathbf{b}$. Conversely, suppose that $A\mathbf{x} = \mathbf{b}$ has a unique solution $\hat{\mathbf{x}}$. If A is singular, $A\mathbf{x} = \mathbf{0}$ has a solution $\mathbf{z} \neq \mathbf{0}$. Let $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$. Clearly, $\mathbf{y} \neq \hat{\mathbf{x}}$ and

$$A\mathbf{y} = A(\hat{\mathbf{x}} + \mathbf{z}) = A\hat{\mathbf{x}} + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

Thus $\mathbf{y}$ is also a solution to $A\mathbf{x} = \mathbf{b}$, which is a contradiction. Therefore, if $A\mathbf{x} = \mathbf{b}$ has a unique solution, A must be nonsingular. $\square$

If A is nonsingular, A is row equivalent to I, so there exist elementary matrices $E_1, \ldots, E_k$ such that

$$E_k E_{k-1} \cdots E_1 A = I$$

Multiplying both sides of this equation on the right by A^{-1}, one obtains

$$E_k E_{k-1} \cdots E_1 I = A^{-1}$$

Thus the same series of elementary row operations that transform a nonsingular matrix A into I will transform I into A^{-1}. This gives us a method for computing A^{-1}. If we augment A by I and perform the elementary row operations that transform A into I on the augmented matrix, then I will be transformed into A^{-1}. That is, the reduced row echelon form of the augmented matrix $(A|I)$ will be $(I|A^{-1})$.

EXAMPLE 7. Compute A^{-1} if

$$\begin{pmatrix} 1 & 4 & 3 \end{pmatrix}$$

SOLUTION

$$\left(\begin{array}{rrr|rrr} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{rrr|rrr} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrr|rrr} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right) \rightarrow \left(\begin{array}{rrr|rrr} 1 & 4 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right) \rightarrow \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right)$$

Thus

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

□

EXAMPLE 8. Solve the system

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 12 \\ -x_1 - 2x_2 \qquad\quad &= -12 \\ 2x_1 + 2x_2 + 3x_3 &= 8 \end{aligned}$$

The coefficient matrix of this system is the matrix A of the last example. The solution to the system then is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 12 \\ -12 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -\frac{8}{3} \end{pmatrix}$$

□

EXERCISES

1. Which of the following are elementary matrices? Classify each elementary matrix by type.

(a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$

2. Find the inverse of each of the matrices in Exercise 1. For each elementary matrix, verify that its inverse is an elementary matrix of the same type.

3. For each of the following pairs of matrices, find an elementary matrix E such that $EA = B$.

(a) $A = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & 2 \\ 5 & 3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{pmatrix}$

(c) $A = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 5 \end{pmatrix}$

4. For each of the following pairs of matrices, find an elementary matrix E such that $AE = B$.

(a) $A = \begin{pmatrix} 4 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & 4 \\ 4 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & 4 \\ 1 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix}$

(c) $A = \begin{pmatrix} 4 & -2 & 3 \\ -2 & 4 & 2 \\ 6 & 1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -2 & 3 \\ -1 & 4 & 2 \\ 3 & 1 & -2 \end{pmatrix}$

5. Given

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{pmatrix}$$

(a) Find an elementary matrix E such that $EA = B$.
(b) Find an elementary matrix F such that $FB = C$.
(c) Is C row equivalent to A? Explain.

6. Given

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{pmatrix}$$

(a) Find elementary matrices E_1, E_2, E_3 such that

$$E_3E_2E_1A = U$$

where U is an upper triangular matrix.

(b) Determine the inverses of E_1, E_2, E_3 and set $L = E_1^{-1}E_2^{-1}E_3^{-1}$. What type of matrix is L? Verify that $A = LU$.

7. Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}$$

(a) Verify that

$$A^{-1} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix}$$

(b) Use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$ for the following choices of $\mathbf{b}$.

(i) $\mathbf{b} = (1, 1, 1)^T$

(ii) $\mathbf{b} = (1, 2, 3)^T$

(iii) $\mathbf{b} = (-2, 1, 0)^T$

8. Find the inverse of each of the following matrices.

(a) $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 6 \\ 3 & 8 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 0 \\ 9 & 3 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & 0 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$

(*) (g) $\begin{pmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{pmatrix}$

9. Given

$$A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

compute A^{-1} and use it to:

(a) Find a 2×2 matrix X such that $AX = B$.

(b) Find a 2×2 matrix Y such that $YA = B$.

10. Given

$$A = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & -2 \\ -6 & 3 \end{pmatrix}$$

Solve each of the following matrix equations.

(a) $AX + B = C$
(b) $XA + B = C$
(c) $AX + B = X$
(d) $XA + C = X$

11. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Show that if $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$, then

$$A^{-1} = \frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

12. Let A be a nonsingular matrix. Show that A^{-1} is also nonsingular and $(A^{-1})^{-1} = A$.

13. Prove that if A is nonsingular, then A^T is nonsingular and

$$(A^T)^{-1} = (A^{-1})^T$$

[***Hint:*** $(AB)^T = B^TA^T$.]

14. Let A be a nonsingular $n \times n$ matrix. Use mathematical induction to prove that A^m is nonsingular and

$$(A^m)^{-1} = (A^{-1})^m$$

for $m = 1, 2, 3, \ldots$.

15. Is the transpose of an elementary matrix an elementary matrix of the same type? Is the product of two elementary matrices an elementary matrix?

16. Let U and R be $n \times n$ upper triangular matrices and set $T = UR$. Show that T is also upper triangular and that $t_{jj} = u_{jj}r_{jj}$ for $j = 1, \ldots, n$.

17. Let A and B be $n \times n$ matrices and let $C = AB$. Prove that if B is singular, then C must be singular.

[***Hint:*** Use Theorem 1.4.3.]

18. Let U be an $n \times n$ upper triangular matrix with nonzero diagonal entries.

(a) Explain why U must be nonsingular.
(b) Explain why U^{-1} must be upper triangular.

19. Let A be a nonsingular $n \times n$ matrix and let B be an $n \times r$ matrix. Show that the reduced row echelon form of $(A|B)$ is $(I|C)$, where $C = A^{-1}B$.

20. In general, matrix multiplication is not commutative (i.e., $AB \neq BA$). However, there are certain special cases where the commutative property does hold. Show that:

(a) If D_1 and D_2 are $n \times n$ diagonal matrices, then $D_1 D_2 = D_2 D_1$.

(b) If A is an $n \times n$ matrix and

$$B = a_0 I + a_1 A + a_2 A^2 + \cdots + a_k A^k$$

where $a_0, a_1, \ldots, a_k$ are scalars, then $AB = BA$.

21. Show that if A is a symmetric nonsingular matrix, then A^{-1} is also symmetric.

22. Prove that if A is row equivalent to B, then B is row equivalent to A.

23. (a) Prove that if A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

(b) Prove that any two nonsingular $n \times n$ matrices are row equivalent.

24. Prove that B is row equivalent to A if and only if there exists a nonsingular matrix M such that $B = MA$.

25. Given a vector $\mathbf{x} \in R^{n+1}$, the $(n+1) \times (n+1)$ matrix V defined by

$$v_{ij} = \begin{cases} 1 & \text{if } j = 1 \\ x_i^{j-1} & \text{for } j = 2, \ldots, n+1 \end{cases}$$

is called the Vandermonde matrix.

(a) Show that if

$$V\mathbf{c} = \mathbf{y}$$

and

$$p(x) = c_1 + c_2 x + \cdots + c_{n+1} x^n$$

then

$$p(x_i) = y_i, \quad i = 1, 2, \ldots, n+1$$

(b) Suppose that $x_1, x_2, \ldots, x_{n+1}$ are all distinct. Show that if $\mathbf{c}$ is a solution to $V\mathbf{x} = \mathbf{0}$, then the coefficients $c_1, c_2, \ldots, c_n$ must all be zero, and hence V must be nonsingular.

PARTITIONED MATRICES

Often it is useful to think of a matrix as being composed of a number of submatrices. A matrix A can be partitioned into smaller matrices by drawing