Name: **Perm number:**

Test 2

Time: 50 minutes

1. Consider the differential equation

$$(1-t)y'' + ty' - y = 0$$

on the interval $1 < t < \infty$

(a) Denote by y_1, y_2 two arbitrary solutions to this equation. Calculate the Wronskian $W(y_1, y_2)$ as follows:

- Write down what does it mean that y_1, y_2 satisfy our equation – you will get two equations.
- Multiply one of them by y_1, the other one by $-y_2$ and add them together.
- What you obtained is an ordinary differential equation in the function $W(y_1, y_2)$; solve it using an integrating factor and get your formula for $W(y_1, y_2)$.

(b) Verify that $y_1(t) = t$ is a solution to our equation.

(c) Now you can use the Wronskian that you found to find y_2! Proceed as follows:

- Write down the definition of Wronskian.
- Now substitute what you already know: in the above definition replace y_1 by t and $W(y_1, y_2)$ by what you found in (a). What you got is another differential equation in the function y_2. Solve it using an integrating factor (Hint: $\int \frac{1-t}{t^2} e^t dt = -e^t/t + C$).

2. In a water purifying plant there are two tanks conntected by one tube, each containing 100 gallons of water. Initially the first tank contains 16 pounds of salt, and the second one 4 pounds of salt. The solution of salt and water flows from the second tank to the first one at a rate of 2 gallons per minute. At the same time, fresh water is pumped into the first tank at the rate of 1 gallon per minute, and into the second tank at a rate of 3 gallons per minute, and the solution is pumped out of the first tank at a rate of 3 gallons per minute, and from the second one at a rate of 1 gallon per minute.

   (a) Picture the two tanks with the 5 described above tubes: one pumping the water in the first tank, one pumping it out, same thing for the second tank, and one more tube connecting both tanks. Indicate the corresponding rates of flow. Now look at your picture and try to convince yourself that it's not as scary as it seems :)

   (b) Denote by $x(t)$ and $y(t)$ amount of salt in the first and the second tank, respectively. Write down the equations describing $x'(t)$ and $y'(t)$. You will obtain a system of two differential equations with two variable functions.

   (c) Write down your system in a matrix form.

   (d) If A denotes the matrix that appers in your equation, write A as a sum $A = B + C$, where B is a diagonal matrix, and C is what's left of A.

   (e) Check that $BtCt = CtBt$.

   (f) Use the definiton of matrix exponentials to compute e^{Bt} and e^{Ct}.

   (g) Use $e^{Bt+Ct} = e^{Bt}e^{Ct}$ to find e^{At}.

   (h) Now write down the formulae for $x(t)$ and $y(t)$ that you've just obtained.