Math 5AI Project 7: Return to Linear Systems

1. (a) Find the eigenvalues for

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

and check that

$$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

are eigenvectors.

(b) Explain why the general solution to $y' = Ay$ is

$$y = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

(c) Find the general solution to $y' = By$ if

$$B = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

2. Consider the system of ODE

$$\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= 4y_1 + y_2 \end{aligned}$$

(a) Calculate y_1'' in terms of y_1 and y_2.

(b) Find a linear combination of y_1'' and y_1' that is a multiple of y_1.

(c) Your answer to (b) gives you a second order constant coefficient linear order ODE in the single function y'. Solve this ODE.

(d) Relate your solution to (c) to your solution to 1 (b).

3. (a) Here you are to reverse the ideas of problem 2. Consider the linear ODE $y'' + 0.1y = 0$. You know how to solve this. Replace this by a system of two first order ODEs

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where $y_1 = y$ and $y_2 = y'$.

(b) If you are allowed to use complex numbers, what are the eigenvalues and the eigenvectors of your system?

(c) Use these complex eigenvectors to sovle this system as you did in problem 1.

(d) How does all this connect to the second order solution of the ODE you started with?

Stability or Instability

A constant solution $y = c$ to a system $y' = f(x)$ is called *stable* if solutions that start sufficiently close to c remain bounded (they may, in fact, approach c). For a 2×2 system you can investigate the stability properties by studying the eigenvalues and thinking about what they mean for the "phase plane". For a system

$$\begin{aligned} y_1' &= ay_1 + by_2 \\ y_2' &= cy_1 + dy_2 \end{aligned}$$

so that

$$y' = Ay \quad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have characterisitc polynomial $C_A(x) = x^2 - (a + d)x + (ab - bc)$. So the eigenvalues are easily calculated and we have a constant solution $y_1 = 0 = y_2$. If for each point $\begin{pmatrix} r \\ s \end{pmatrix}$ in the y_1-y_2 plane we plot the direction of $A\begin{pmatrix} r \\ s \end{pmatrix}$ we obtain a vector field of this system in its phase-plane (more or less as we did for single first-order ODEs). This indicates the trajectories of solutions as they evolve over the independent variable (time, perhaps) and you can investigate whether they approach or retreat from the constant solution.

4. Here you are to figure out what is meant in the above paragraph with a very simple example. Suppose that

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

The solutions to $y' = Ay$ are instantly written down. Write down three solutions that satisfy
(i) $y_1(0) = 0$ and $y_2(0) = 0$
(ii) $y_1(0) = 1$ and $y_2(0) = 0$
(iii) $y_1(0) = 1$ and $y_2(0) = 1$
Graph the vector field of this system in its phase-plane and compare it to the graphs of the solutions to the ODE with these initial conditions. *In making this graph, don't take a lot of time calculating values–you should be able to graph it knowing what the eigenvectors and eigenvalues for A are.* What information does this phase-plane graph give you?

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5. Make quick sketches for the vector field of the phase-plane for $y' = A_i y$ $i = 1, 2, 3$ where A_i is given by

$$A_1 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \quad , \quad A_2 = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$$

For this, the eigenvalues of A_1 are -1 and -3, the eigenvalues of A_2 are 4 and 1 and the eigenvalue sof A_4 are 3 and -1. What information does this phase-plane graph give you in each case about the nature of solutions? Discuss this.

6. If the eigenvalues of a matrix A are complex, say $\alpha \pm \beta i$ then there are three cases. If $\alpha = 0$ you get loops or a neutrally stable set of solutions. If $\alpha < 0$ then you get an attracting spiral, or a spiral sink, and if $\alpha > 0$ then you get an repelling spiral, or a spiral source. In terms of the phase plane, explain what these terminologies must mean. (Note, because you are graphing real parts of y_1 and y_2 you won't see these eigenvalues on the phase plane!).