Take home final

1. **Bases.** A basis of an n-dimensional vector space is a set of vectors $v_1, \ldots, v_n$ which are linearly independent. Verify that the following sets of vectors are bases of indicated spaces:
   (a) $\mathbb{R}^3$, $\alpha_1 = [1,\ 0,\ 1]$, $\alpha_2 = [1,\ 1,\ 0]$, $\alpha_3 = [1,\ 1,\ 1]$,
   (b) $\mathbb{R}^3$, $\alpha_1 = [2,\ 1,\ 1]$, $\alpha_2 = [1,\ 3,\ 1]$, $\alpha_3 = [1,\ 1,\ 4]$,
   (c) $\mathbb{R}^2_2$, $\alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$,

2. **Linear maps as matrices.** Suppose that $T : V \to W$ is a linear map, that V is a $n-$dimensional vector space with a basis $\alpha_1, \ldots, \alpha_n$, and that W is a $m-$dimensional vector space with a basis $\beta_1, \ldots, \beta_m$. We build the matrix $A = [a_{ij}]$ of T in the bases $\alpha_1, \ldots, \alpha_n$ by definiting its entries as follows:

$$T(\alpha_i) = a_{i1}\beta_1 + a_{i2}\beta_2 + \ldots + a_{im}\beta_m.$$

   In other words, coefficients $a_{i1}, a_{i2}, \ldots, a_{im}$ of the combination of the image of α_i form i-th column of the matrix A. We choose

$$[1,1,0], [-1,2,1], [1,0,1]$$

   to be the basis of $\mathbb{R}^3$, and

$$[2,1,0,1], [-1,1,-1,1], [0,1,2,0], [-2,0,0,0]$$

   to be the basis of $\mathbb{R}^4$. Verify that the following maps are linear and find their matrices in the respective bases:
   (a) $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi([x,y,z]) = [x+z, 2x+z, 3x-y+z]$,
   (b) $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi([x,y,z]) = [x-y+z, y, z]$,
   (c) $\varphi : \mathbb{R}^4 \to \mathbb{R}^3$, $\varphi([x,y,z,t]) = [x-y+2t, 2x+3y+5z-t, x+z-t]$,

3. **Compositions of linear maps.** Let $T : V \to V$ and $S : V \to$ be two linear maps (we assume for simplicity that both the domain and codomain of these maps is the same space V). Take three bases of the space V, say $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Let A be the matrix of T in the bases $\mathcal{A}$ and $\mathcal{B}$, and B the matrix of S in the bases $\mathcal{B}$ and $\mathcal{C}$. Prove that the matrix of $S \circ T$ in the bases $\mathcal{A}$ and $\mathcal{C}$ is BA. Stick to the case when $\dim V = 2$.

4. **Change of bases vs. change of matrices.**
   (a) Let $T : V \to V$ be a linear map whose matrix in the bases $\mathcal{A}$ in $\mathcal{B}$ is equal to A. Use the result from the previous problem to show that the matrix of T^{-1} in the bases $\mathcal{B}$ and $\mathcal{A}$ is A^{-1}.
   (b) That the vector v has coordinates $[1,2,3]$ means that it can be represented as the following linear combination of basic vectors $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$:

$$[1,2,3] = 1 \cdot [1,0,0] + 2 \cdot [0,1,0] + 3 \cdot [0,0,1].$$

   Suppose we want to write it as a linear combination of vectors from another basis, say $[1,1,0]$, $[1,0,1]$, $[0,1,1]$. Say that a, b, c are the resulting coefficients in the linear combination (find them!). Now write down the matrix of the identity mapping $Id : \mathbb{R}^3 \to \mathbb{R}^3$ in the bases $[1,0,0]$, $[0,1,0]$, $[0,0,1]$ and $[1,1,0]$, $[1,0,1]$, $[0,1,1]$, call it P. Show that $[a,b,c]^T = A \cdot [1,2,3]^T$. What will be the new "coordinates" in the basis $[1,1,0]$, $[1,0,1]$, $[0,1,1]$ of the vector $[77,-15,99]$? Of an arbitrary vector $[x,y,z]$?
   (c) Now suppose that a mean and vicious math professor gives you problems with all vectors written in "coordinates" in the basis $[1,1,0]$, $[1,0,1]$, $[0,1,1]$. Show that multiplying them by P^{-1} gives you vectors in "normal" coordinates.
   (d) Combine (a), (b), (c) and the previous problem to show the following theorem: let $T : V \to V$ be a linear map whose matrix in the basis $\mathcal{A}$ (i.e. we treat $\mathcal{A}$ as the basis of both domain and codomain) is equal to A, let P be the matrix of the identity map in the bases $\mathcal{A}$ and $\mathcal{B}$, let B be the matrix of T in the basis $\mathcal{B}$. Then:

$$B = PAP^{-1}.$$

5. **Eigenvalues and eigenvectors.** If A is a matrix then a number λ is called an eigenvalue if for some v, which is then called an eigenvector, $Av = \lambda v$.

(a) How can we find eigenvalues? Suppose we are given a matrix A; we are looking for a number λ and a vector v such that

$$Av = \lambda v$$

or, equvalently:

$$Av = \lambda I v$$

where I denotes the identity matrix. Moving everything to the left side of the equation and factoring out v gives

$$(A - \lambda I)v = 0.$$

This is a system of equations with the parameter λ, where variables are coordinates of the vector v, right? It has one trivial solution, when we simply take $v = 0$, but we are more interested in nontrivial solutions. Think of the matrix $A - \lambda I$. What condition shall we impose on this matrix in order to guarantee existence of nontrivial solutions to our system? (hint: think determinants). Does it give you a method of finding eigenvalues? (it sure does! – just name it). Once you found an eigenvalue, finding corresponding eigenvectors is easy – you just solve system of linear equations $Av = \lambda v$, where now λ is a given number.

(b) Find eigenvalues and corresponding eigenvectors for the following matrices:

$$\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

6. **Diagonalization.** Now the real fun begins! Suppose we have a square $n \times n$ matrix A, and we want to decompose it as PBP^{-1}, where B is diagonal.

   (a) Think of A as of a matrix of some linear mapping T in the “cannonical” basis $[1, 0, 0, \ldots, 0]$, $[0, 1, 0, \ldots, 0]$, $[0, 0, 1, \ldots, 0]$, …, $[0, 0, 0, \ldots, 1]$. Suppose that we found n eigenvalues for A, call them $\lambda_1, \lambda_2, \ldots, \lambda_n$, and that the corresponding eigenvectors, call them $v_1, v_2, \ldots, v_n$, form a basis. Find the matrix B of T in the basis $v_1, v_2, \ldots, v_n$.

   (b) Write down the matrix of the identity map in the bases $v_1, v_2, \ldots, v_n$ and the “cannonical” basis; call it P.

   (c) You scored big time! Use what you already have to show that $A = PBP^{-1}$.

   (d) Write down the diagonal decompositions PBP^{-1} for matrices from the previous problem.

7. **Cleaning up and filling in gaps.** Go back to Project 4 (Mixing salt and water) and fill in the missing part of your solution to Problem 8.

8. **Life is brutal...** So, now you think you can solve every single problem involving systems of linear differential equations with constant coefficients, eh? Not to let you down after all the work you put in this project, but... think what can go wrong. First of all, would you be able to find eigenvalues of a matrix, say, 100×100? Why? Why not? Use Wikipedia to learn something about Abel-Ruffini theorem. Secondly, who said that eigenvalues, or, at least, real eigenvalues, exist for every single matrix? Can you, perhaps, give some examples (if you can, they will count as bonus points!)? But don’t despair – diagonal decomposition of a matrix is just one of many decompositions that help us deal with systems of differential equations, dynamical systems etc. In fact, what you’ve just seen, is merely a tip of an iceberg – but hey, that’s a good reason for taking Math 5BI!

9. **Kaplansky’s principle.** Verify that the previous problem satisfies the following Kaplansky’s [1] principle: “As the work comes to a close the theorems fade away in favor of hand-waving on a large scale. But that is the way every course should finish”.

[1] Irving Kaplansky (1917-2006) – was a Canadian mathematician, born to Polish parents. He worked for most of his life at the University of Chicago, his research interests included commutative algebra and geometry. For more details see `http://www-history.mcs.st-andrews.ac.uk/Biographies/Kaplansky.html`