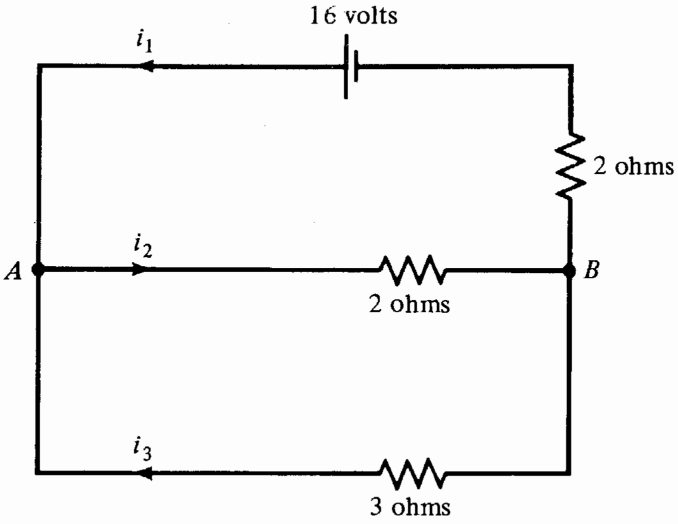
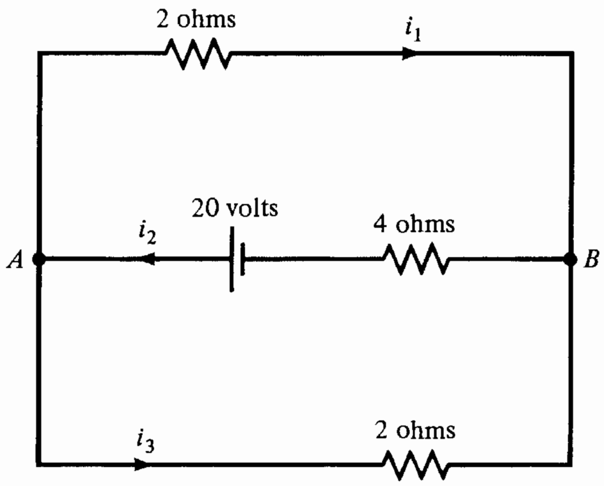
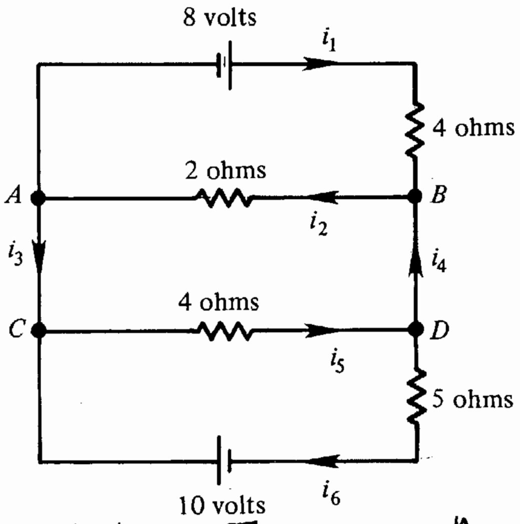
(a)




(b)




(c)




ASSIGNMENT 3 due date: Tuesday, August 1, 2

## 3 MATRIX ALGEBRA

In this section we define arithmetic operations with matrices and look at some of their algebraic properties. Matrices are one of the most powerful tools in mathematics. To use matrices effectively, we must be adept at matrix arithmetic.

The entries of a matrix are called *scalars*. They are usually either real or complex numbers. For the most part we will be working with matrices whose entries are real numbers. Throughout the first five chapters of the book the reader may assume that the term *scalar* refers to a real number. However, in Chapter 6 there will be occasions when we will use the set of complex numbers as our scalar field.

If we wish to refer to matrices without specifically writing out all their entries, we will use capital letters $A$, $B$, $C$, and so on. In general, $a_{ij}$ will denote the entry of the matrix $A$ that is in the $i$th row and the $j$th column. Thus if $A$ is an $m \times n$ matrix, then

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

We will sometimes shorten this to $A = (a_{ij})$. Similarly, a matrix $B$ may be referred to as $(b_{ij})$, a matrix $C$ as $(c_{ij})$, and so on.

## EQUALITY

***Definition.*** Two $m \times n$ matrices $A$ and $B$ are said to be **equal** if $a_{ij} = b_{ij}$ for each $i$ and $j$.

## SCALAR MULTIPLICATION

If $A$ is a matrix and $\alpha$ is a scalar, then $\alpha A$ is the matrix formed by multiplying each of the entries of $A$ by $\alpha$. For example, if

$$A = \begin{pmatrix} 4 & 8 & 2 \\ 6 & 8 & 10 \end{pmatrix}$$

then

$$\tfrac{1}{2}A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 4 & 5 \end{pmatrix} \quad \text{and} \quad 3A = \begin{pmatrix} 12 & 24 & 6 \\ 18 & 24 & 30 \end{pmatrix}$$

## MATRIX ADDITION

If $A = (a_{ij})$ and $B = (b_{ij})$ are both $m \times n$ matrices, then the sum $A + B$ is the $m \times n$ matrix whose $ij$th entry is $a_{ij} + b_{ij}$ for each ordered pair $(i, j)$. For example,

$$\begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 3 \\ 5 & 7 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix} + \begin{pmatrix} -8 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ 10 \end{pmatrix}$$

If we define $A - B$ to be $A + (-1)B$, then it turns out that $A - B$ is formed by subtracting the corresponding entry of $B$ from each entry of $A$. Thus

$$\begin{aligned}\begin{pmatrix}2 & 4\\3 & 1\end{pmatrix} - \begin{pmatrix}4 & 5\\2 & 3\end{pmatrix} &= \begin{pmatrix}2 & 4\\3 & 1\end{pmatrix} + (-1)\begin{pmatrix}4 & 5\\2 & 3\end{pmatrix}\\ &= \begin{pmatrix}2 & 4\\3 & 1\end{pmatrix} + \begin{pmatrix}-4 & -5\\-2 & -3\end{pmatrix}\\ &= \begin{pmatrix}2-4 & 4-5\\3-2 & 1-3\end{pmatrix}\\ &= \begin{pmatrix}-2 & -1\\1 & -2\end{pmatrix}\end{aligned}$$

If $O$ represents a matrix, with the same dimensions as $A$, whose entries are all 0, then

$$A + O = O + A = A$$

That is, the zero matrix acts as an additive identity on the set of all $m \times n$ matrices. Furthermore, each $m \times n$ matrix $A$ has an additive inverse. Indeed,

$$A + (-1)A = O = (-1)A + A$$

It is customary to denote the additive inverse by $-A$. Thus

$$-A = (-1)A$$

## MATRIX MULTIPLICATION

We have yet to define the most important operation, the multiplication of two matrices. Much of the motivation behind the definition comes from the applications to linear systems of equations. If we have a system of one linear equation in one unknown, it can be written in the form

$$ax = b \tag{1}$$

We generally think of $a$, $x$, and $b$ as being scalars; however, they could also be treated as $1 \times 1$ matrices. More generally, given an $m \times n$ linear system

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\ &\vdots\\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

it is desirable to write the system in a form similar to (1), that is, as a matrix equation

$$AX = B$$

where $A = (a_{ij})$ is known, $X$ is an $n \times 1$ matrix of unknowns, and $B$ is an $m \times 1$ matrix representing the right-hand side of the system. Thus we set

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

and

$$(2) \qquad AX = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

Given an $m \times n$ matrix $A$ and an $n \times 1$ matrix $X$ it is possible to compute a product $AX$ by (2). The product $AX$ will be an $m \times 1$ matrix. The rule for determining the $i$th entry of $AX$ is

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

Note that the $i$th entry is determined using only the $i$th row of $A$. The entries in that row are paired off with the corresponding entries of $X$ and multiplied. The $n$ products are then summed. Those readers familiar with dot products will recognize this as simply the dot product of the $n$-tuple corresponding to the $i$th row of $A$ with the $n$-tuple corresponding to the matrix $X$.

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

In order to pair off the entries in this way, the number of columns of $A$ must equal the number of rows of $X$. The entries of $X$ can be either scalars or unknowns having scalar values.

**EXAMPLE 1**

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 5 & 3 & 7 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$AX = \begin{pmatrix} 4x_1 + 2x_2 + x_3 \\ 5x_1 + 3x_2 + 7x_3 \end{pmatrix}$$

□

**EXAMPLE 2**

$$A = \begin{pmatrix} -3 & 1 \\ 2 & 5 \\ 4 & 2 \end{pmatrix}, \qquad X = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$AX = \begin{pmatrix} -3 \cdot 2 + 1 \cdot 4 \\ 2 \cdot 2 + 5 \cdot 4 \\ 4 \cdot 2 + 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 24 \\ 16 \end{pmatrix}$$

□

**EXAMPLE 3.** Write the following system of equations as a matrix equation $AX = B$.

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 5 \\ x_1 - 2x_2 + 5x_3 &= -2 \\ 2x_1 + x_2 - 3x_3 &= 1 \end{aligned}$$

SOLUTION

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & -2 & 5 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

□

More generally, it is possible to multiply a matrix $A$ times a matrix $B$ if the number of columns of $A$ equals the number of rows of $B$. The first column of the product is determined by the first column of $B$, the second column by the second column of $B$, and so on. Thus, to determine the $(i, j)$ entry of the product $AB$, we use the entries of the $i$th row of $A$ and the $j$th column of $B$.

***Definition.*** If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times r$ matrix, then the product $AB = C = (c_{ij})$ is the $m \times r$ matrix whose entries are defined by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

What this definition says is that to find the $ij$th element of the product, you take the $i$th row of $A$ and the $j$th column of $B$, multiply the corresponding elements pairwise, and add the resulting numbers.

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}$$

## NOTATIONAL RULES

Just as in ordinary algebra, if an expression involves both multiplication and addition and there are no parentheses to indicate the order of the operations, multiplications are carried out before additions. This is true for both scalar and matrix multiplications. For example, if

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix}$$

then

$$A + BC = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 7 & 7 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 11 \\ 0 & 6 \end{pmatrix}$$

and

$$3A + B = \begin{pmatrix} 9 & 12 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 15 \\ 5 & 7 \end{pmatrix}$$

## ALGEBRAIC RULES

The following theorem provides some useful rules for doing matrix arithmetic.

**Theorem 1.3.1.** *Each of the following statements is valid for any scalars $\alpha$ and $\beta$ and for any matrices $A$, $B$, and $C$ for which the indicated operations are defined.*

(1) $A + B = B + A$
(2) $(A + B) + C = A + (B + C)$
(3) $(AB)C = A(BC)$
(4) $A(B + C) = AB + AC$
(5) $(A + B)C = AC + BC$
(6) $(\alpha\beta)A = \alpha(\beta A)$
(7) $\alpha(AB) = (\alpha A)B = A(\alpha B)$
(8) $(\alpha + \beta)A = \alpha A + \beta A$
(9) $\alpha(A + B) = \alpha A + \alpha B$

We will prove two of the rules and leave the rest for the reader to verify.

*Proof of (4).* Assume that $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ and $C = (c_{ij})$ are both $n \times r$ matrices. Let $D = A(B + C)$ and $E = AB + AC$. It follows that

$$d_{ij} = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$$

and

$$e_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$

But

$$\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$

so that $d_{ij} = e_{ij}$ and hence $A(B + C) = AB + AC$. □

*Proof of (3).* Let $A$ be an $m \times n$ matrix, $B$ an $n \times r$ matrix, and $C$ an $r \times s$ matrix. Let $D = AB$ and $E = BC$. We must show that $DC = AE$. By the definition of matrix multiplication,

$$d_{il} = \sum_{k=1}^{n} a_{ik}b_{kl} \qquad \text{and} \qquad e_{kj} = \sum_{l=1}^{r} b_{kl}c_{lj}$$

The $ij$th term of $DC$ is

$$\sum_{l=1}^{r} d_{il}c_{lj} = \sum_{l=1}^{r} \left( \sum_{k=1}^{n} a_{ik}b_{kl} \right) c_{lj}$$

and the $ij$th entry of $AE$ is

$$\sum_{k=1}^{n} a_{ik}e_{kj} = \sum_{k=1}^{n} a_{ik} \left( \sum_{l=1}^{r} b_{kl}c_{lj} \right)$$

Since

$$\sum_{l=1}^{r} \left( \sum_{k=1}^{n} a_{ik}b_{kl} \right) c_{lj} = \sum_{l=1}^{r} \sum_{k=1}^{n} a_{ik}b_{kl}c_{lj} = \sum_{k=1}^{n} a_{ik} \left( \sum_{l=1}^{r} b_{kl}c_{lj} \right)$$

it follows that

$$(AB)C = DC = AE = A(BC)$$ □

The arithmetic rules given in Theorem 1.3.1 seem quite natural since they are similar to the rules we use with real numbers. However, there are some important differences between the rules for matrix arithmetic and those for real number arithmetic. In particular, multiplication of real numbers is commutative; however, we saw in Example 6 that matrix multiplication is not commutative. This difference warrants special emphasis.

> **Warning:** In general, $AB \neq BA$.
> Matrix multiplication is *not* commutative.

Some of the other differences between matrix arithmetic and real number arithmetic are illustrated in Exercises 13, 14, and 15.

EXAMPLE 7. If

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}, \qquad \text{and} \qquad C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

verify that $A(BC) = (AB)C$ and $A(B + C) = AB + AC$.

SOLUTION

$$A(BC) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 16 & 11 \end{pmatrix}$$

$$(AB)C = \begin{pmatrix} -4 & 5 \\ -6 & 11 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 16 & 11 \end{pmatrix}$$

Thus

$$A(BC) = \begin{pmatrix} 6 & 5 \\ 16 & 11 \end{pmatrix} = (AB)C$$

$$A(B + C) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 5 & 15 \end{pmatrix}$$

$$AB + AC = \begin{pmatrix} -4 & 5 \\ -6 & 11 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ 11 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 5 & 15 \end{pmatrix}$$

Therefore,

$$A(B + C) = AB + AC$$

□

**Notation.** Since $(AB)C = A(BC)$, one may simply omit the parentheses and write $ABC$. The same is true for a product of four or more matrices. In the case where an $n \times n$ matrix is multiplied by itself a number of times, it is convenient to use exponential notation. Thus if $k$ is a positive integer, then

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}}$$

**EXAMPLE 8.** If

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

then

$$A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$A^3 = AAA = AA^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

and in general

$$A^n = \begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}$$

□

(a) Determine the adjacency matrix $A$ of the graph.

(b) Compute $A^2$. What do the entries in the first row of $A$ tell you about walks of length 2 that start from $V_1$?

(c) Compute $A^3$. How many walks of length 3 are there from $V_2$ to $V_4$? How many walks of length less than or equal to 3 are there from $V_2$ to $V_4$?

**23.** Let $A$ be a $2 \times 2$ matrix with $a_{11} \neq 0$ and let $\alpha = a_{21}/a_{11}$. Show that $A$ can be factored into a product of the form

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & b \end{pmatrix}$$

What is the value of $b$?

## SPECIAL TYPES OF MATRICES

In this section we look at special types of $n \times n$ matrices, such as triangular matrices, diagonal matrices, and elementary matrices. These special types of matrices play an important role in the solution of matrix equations. We begin by considering a special matrix $I$ that acts like a multiplicative identity, that is,

$$IA = AI = A$$

for any $n \times n$ matrix $A$. We also discuss the existence and computation of multiplicative inverses.

### THE IDENTITY MATRIX

One very important matrix is the $n \times n$ matrix $I$ with 1's on the diagonal and 0's off the diagonal. Thus $I = (\delta_{ij})$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

If $A$ is any $n \times n$ matrix, $AI = IA = A$. The matrix $I$ acts as an identity for the multiplication of $n \times n$ matrices and consequently is referred to as the *identity matrix*. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{pmatrix}$$

and

$$\begin{pmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{pmatrix}$$

In general, if $B$ is any $m \times n$ matrix and $C$ is any $n \times r$ matrix, then

$$BI = B \qquad \text{and} \qquad IC = C$$

**Notation.** The set of all $n$-tuples of real numbers is called *Euclidean n-space* and is usually denoted by $R^n$. The elements of $R^n$ are called *vectors*. Note, however, that the solution to the matrix equation $AX = B$ will be an $n \times 1$ matrix rather than an $n$-tuple. In general, when working with matrix equations it is more convenient to think of $R^n$ as consisting of column vectors ($n \times 1$ matrices) rather than row vectors ($1 \times n$ matrices). The standard notation for a column vector is a boldface lowercase letter.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \mathbf{x}^T = (x_1, \ldots, x_n)$$

column vector row vector

Following this convention, we will use the notation $A\mathbf{x} = \mathbf{b}$, rather than $AX = B$, to represent a linear system of equations.

Given an $m \times n$ matrix $A$, it is often necessary to refer to a particular row or column. The $i$th row vector of $A$ will be denoted by $\mathbf{a}(i, :)$ and the $j$th column vector will be denoted by $\mathbf{a}(:, j)$. In general we will be working primarily with column vectors. Consequently, we will use the shorthand notation $\mathbf{a}_j$ in place of $\mathbf{a}(:, j)$. Since references to row vectors are far less frequent, we will not use any shorthand notation for row vectors. In summation, if $A$ is an $m \times n$ matrix, then the row vectors of $A$ are given by

$$\mathbf{a}(i, :) = (a_{i1}, a_{i2}, \ldots, a_{in}) \qquad i = 1, \ldots, m$$

and the column vectors are given by

$$\mathbf{a}_j = \mathbf{a}(:, j) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \qquad j = 1, \ldots, n$$

Similarly, if $B$ is an $n \times r$ matrix, then $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r)$. The only exception to this notation is in the case of the identity matrix $I$. The standard notation for the $j$th column vector of $I$ is $\mathbf{e}_j$ rather than $\mathbf{i}_j$. Thus the $n \times n$ identity matrix can be written

$$I = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$$

## DIAGONAL AND TRIANGULAR MATRICES

An $n \times n$ matrix $A$ is said to be *upper triangular* if $a_{ij} = 0$ for $i > j$ and *lower triangular* if $a_{ij} = 0$ for $i < j$. Also, $A$ is said to be *triangular* if it is either upper triangular or lower triangular. For example, the $3 \times 3$ matrices

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 1 & 4 & 3 \end{pmatrix}$$

are both triangular. The first is upper triangular and the second is lower triangular.

A triangular matrix may have 0's on the diagonal. However, for a linear system $A\mathbf{x} = \mathbf{b}$ to be in triangular form, the coefficient matrix $A$ must be triangular with nonzero diagonal entries.

An $n \times n$ matrix $A$ is *diagonal* if $a_{ij} = 0$ whenever $i \neq j$. The matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are all diagonal. A diagonal matrix is both upper triangular and lower triangular.

## MATRIX INVERSION

***Definition.*** An $n \times n$ matrix $A$ is said to be **nonsingular** or **invertible** if there exists a matrix $B$ such that $AB = BA = I$. The matrix $B$ is said to be a multiplicative inverse of $A$.

If $B$ and $C$ are both multiplicative inverses of $A$, then

$$B = BI = B(AC) = (BA)C = IC = C$$

Thus a matrix can have at most one multiplicative inverse. We will refer to the multiplicative inverse of a nonsingular matrix $A$ as simply the *inverse* of $A$ and denote it by $A^{-1}$.

**EXAMPLE 1.** The matrices

$$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{pmatrix}$$

are inverses of each other, since

$$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

**EXAMPLE 2.** The triangular matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

are inverses, since

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

□

**EXAMPLE 3.** The matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

has no inverse. Indeed, if $B$ is any $2 \times 2$ matrix, then

$$BA = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b_{11} & 0 \\ b_{21} & 0 \end{pmatrix}$$

Thus $BA$ cannot equal $I$. □

***Definition.*** An $n \times n$ matrix is said to be **singular** if it does not have a multiplicative inverse.

## EQUIVALENT SYSTEMS

Given an $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$, we can obtain an equivalent system by multiplying both sides of the equation by a nonsingular $m \times m$ matrix $M$.

(1) $$A\mathbf{x} = \mathbf{b}$$

(2) $$MA\mathbf{x} = M\mathbf{b}$$

Clearly, any solution to (1) will also be a solution to (2). On the other hand, if $\hat{\mathbf{x}}$ is a solution to (2), then

$$M^{-1}(MA\hat{\mathbf{x}}) = M^{-1}(M\mathbf{b})$$
$$A\hat{\mathbf{x}} = \mathbf{b}$$

so the two systems are equivalent.

To obtain an equivalent system that is easier to solve, we can apply a sequence of nonsingular matrices $E_1, \ldots, E_k$ to both sides of the equation

$A\mathbf{x} = \mathbf{b}$ to obtain a simpler system

$$U\mathbf{x} = \mathbf{c}$$

where $U = E_k \cdots E_1 A$ and $\mathbf{c} = E_k \cdots E_2 E_1 \mathbf{b}$. The new system will be equivalent to the original provided that $M = E_k \cdots E_1$ is nonsingular. However, $M$ is the product of nonsingular matrices. The following theorem shows that any product of nonsingular matrices is nonsingular.

***Theorem 1.4.1.*** *If $A$ and $B$ are nonsingular $n \times n$ matrices, then $AB$ is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.*

*Proof*

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}B = I$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I \qquad \square$$

It follows by induction that if $E_1, \ldots, E_k$ are all nonsingular, then the product $E_1E_2 \cdots E_k$ is nonsingular and

$$(E_1E_2 \cdots E_k)^{-1} = E_k^{-1} \cdots E_2^{-1}E_1^{-1}$$

We will show next that any of the three elementary row operations can be accomplished by multiplying $A$ on the left by a nonsingular matrix.

## ELEMENTARY MATRICES

A matrix obtained from the identity matrix $I$ by the performance of one elementary row operation is called an *elementary* matrix.

There are three types of elementary matrices corresponding to the three types of elementary row operations.

### Type I

An elementary matrix of type I is a matrix obtained by interchanging two rows of $I$.

**EXAMPLE 4.** Let

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$E_1$ is an elementary matrix of type I, since it was obtained by interchanging the first two rows of $I$. Let $A$ be a $3 \times 3$ matrix.

$$E_1A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$AE_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{pmatrix}$$

Multiplying $A$ on the left by $E_1$ interchanges the first and second rows of $A$. Right multiplication of $A$ by $E_1$ is equivalent to the elementary column operation of interchanging the first and second columns. □

### Type II

An elementary matrix of type II is a matrix obtained by multiplying a row of $I$ by a nonzero constant.

**EXAMPLE 5**

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

is an elementary matrix of type II.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 3a_{13} \\ a_{21} & a_{22} & 3a_{23} \\ a_{31} & a_{32} & 3a_{33} \end{pmatrix}$$

Multiplication on the left by $E_2$ performs the elementary row operation of multiplying the third row by 3, while multiplication on the right by $E_2$ performs the elementary column operation of multiplying the third column by 3. □

### Type III

An elementary matrix of type III is a matrix obtained from $I$ by adding a multiple of one row to another row.

**EXAMPLE 6**

$$E_3 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is an elementary matrix of type III. If $A$ is a $3 \times 3$ matrix, then

$$E_3A = \begin{pmatrix} a_{11} + 3a_{31} & a_{12} + 3a_{32} & a_{13} + 3a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$AE_3 = \begin{pmatrix} a_{11} & a_{12} & 3a_{11} + a_{13} \\ a_{21} & a_{22} & 3a_{21} + a_{23} \\ a_{31} & a_{32} & 3a_{31} + a_{33} \end{pmatrix}$$

Multiplication on the left by $E_3$ adds 3 times the third row to the first row. Multiplication on the right adds 3 times the first column to the third column. □

In general, suppose that $E$ is an $n \times n$ elementary matrix. We can think of $E$ as being obtained from $I$ by either a row operation or a column operation. If $A$ is an $n \times r$ matrix, premultiplying $A$ by $E$ has the effect of performing that same row operation on $A$. If $B$ is an $m \times n$ matrix, postmultiplying $B$ by $E$ is equivalent to performing that same column operation on $B$.

***Theorem 1.4.2.*** *If $E$ is an elementary matrix, then $E$ is nonsingular and $E^{-1}$ is an elementary matrix of the same type.*

*Proof.* If $E$ is the elementary matrix of type I formed from $I$ by interchanging the $i$th and $j$th rows, then $E$ can be transformed back into $I$ by interchanging these same rows again. Thus $EE = I$ and hence $E$ is its own inverse. If $E$ is the elementary matrix of type II formed by multiplying the $i$th row of $I$ by a nonzero scalar $\alpha$, then $E$ can be transformed into the identity by multiplying either its $i$th row or its $i$th column by $1/\alpha$. Thus

$$E^{-1} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & O & \\ & & 1 & & & & \\ & & & 1/\alpha & & & \\ & & & & 1 & & \\ & O & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \quad i\text{th row}$$

Finally, suppose that $E$ is the elementary matrix of type III formed from $I$ by adding $m$ times the $i$th row to the $j$th row.

$$E = \begin{pmatrix} 1 & & & & & & \\ \vdots & \ddots & & & & O & \\ 0 & \cdots & 1 & & & & \\ \vdots & & & \ddots & & & \\ 0 & \cdots & m & \cdots & 1 & & \\ \vdots & & & & & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \quad \begin{matrix} \\ \\ i\text{th row} \\ \\ j\text{th row} \\ \\ \\ \end{matrix}$$

$E$ can be transformed back into $I$ by either subtracting $m$ times the $i$th row column. Thus

$$E^{-1} = \begin{pmatrix} 1 & & & & & & \\ \vdots & \ddots & & & & O & \\ 0 & \cdots & 1 & & & & \\ \vdots & & & \ddots & & & \\ 0 & \cdots & -m & \cdots & 1 & & \\ \vdots & & & & & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}$$

□

***Definition.*** A matrix $B$ is **row equivalent** to $A$ if there exists a finite sequence $E_1, E_2, \ldots, E_k$ of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_1 A$$

In other words, $B$ is row equivalent to $A$ if $B$ can be obtained from $A$ by a finite number of row operations. In particular, two augmented matrices $(A \mid \mathbf{b})$ and $(B \mid \mathbf{c})$ are row equivalent if and only if $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{c}$ are equivalent systems.

The following properties of row equivalent matrices are consequences of Theorem 1.4.2.

(i) If $A$ is row equivalent to $B$, then $B$ is row equivalent to $A$.
(ii) If $A$ is row equivalent to $B$, and $B$ is row equivalent to $C$, then $A$ is row equivalent to $C$.

The details of the proofs of (i) and (ii) are left as an exercise for the reader.

***Theorem 1.4.3.*** *Let $A$ be an $n \times n$ matrix. The following are equivalent:*

(a) *$A$ is nonsingular.*
(b) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$.*
(c) *$A$ is row equivalent to $I$.*

*Proof.* We prove first that statement (a) implies statement (b). If $A$ is nonsingular and $\hat{\mathbf{x}}$ is a solution to $A\mathbf{x} = \mathbf{0}$, then

$$\hat{\mathbf{x}} = I\hat{\mathbf{x}} = (A^{-1}A)\hat{\mathbf{x}} = A^{-1}(A\hat{\mathbf{x}}) = A^{-1}\mathbf{0} = \mathbf{0}$$

Thus $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Next we show that statement (b) implies statement (c). If we use elementary row operations, the system can be transformed into the form $U\mathbf{x} = \mathbf{0}$, where $U$ is in row echelon form. If one of the diagonal elements of $U$ were 0, the last row of $U$ would consist entirely of 0's. But then $A\mathbf{x} = \mathbf{0}$ would be equivalent to a system with more unknowns than equations and hence by Theorem 1.2.1 would have a nontrivial solution. Thus $U$ must be a triangular matrix with diagonal elements all equal to 1. It follows then that $I$ is the reduced row echelon form of $A$ and hence $A$ is row

Finally, we will show that statement (c) implies statement (a). If $A$ is row equivalent to $I$, there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$A = E_k E_{k-1} \cdots E_1 I = E_k E_{k-1} \cdots E_1$$

But since $E_i$ is invertible, $i = 1, \ldots, k$, the product $E_k E_{k-1} \cdots E_1$ is also invertible. Hence $A$ is nonsingular and

$$A^{-1} = (E_k E_{k-1} \cdots E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1} \qquad \square$$

**Corollary 1.4.4.** *The system of n linear equations in n unknowns $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $A$ is nonsingular.*

*Proof.* If $A$ is nonsingular, then $A^{-1}\mathbf{b}$ is the only solution to $A\mathbf{x} = \mathbf{b}$. Conversely, suppose that $A\mathbf{x} = \mathbf{b}$ has a unique solution $\hat{\mathbf{x}}$. If $A$ is singular, $A\mathbf{x} = \mathbf{0}$ has a solution $\mathbf{z} \neq \mathbf{0}$. Let $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$. Clearly, $\mathbf{y} \neq \hat{\mathbf{x}}$ and

$$A\mathbf{y} = A(\hat{\mathbf{x}} + \mathbf{z}) = A\hat{\mathbf{x}} + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

Thus $\mathbf{y}$ is also a solution to $A\mathbf{x} = \mathbf{b}$, which is a contradiction. Therefore, if $A\mathbf{x} = \mathbf{b}$ has a unique solution, $A$ must be nonsingular. $\square$

If $A$ is nonsingular, $A$ is row equivalent to $I$, so there exist elementary matrices $E_1, \ldots, E_k$ such that

$$E_k E_{k-1} \cdots E_1 A = I$$

Multiplying both sides of this equation on the right by $A^{-1}$, one obtains

$$E_k E_{k-1} \cdots E_1 I = A^{-1}$$

Thus the same series of elementary row operations that transform a nonsingular matrix $A$ into $I$ will transform $I$ into $A^{-1}$. This gives us a method for computing $A^{-1}$. If we augment $A$ by $I$ and perform the elementary row operations that transform $A$ into $I$ on the augmented matrix, then $I$ will be transformed into $A^{-1}$. That is, the reduced row echelon form of the augmented matrix $(A|I)$ will be $(I|A^{-1})$.

**EXAMPLE 7.** Compute $A^{-1}$ if

$$\begin{pmatrix} 1 & 4 & 3 \end{pmatrix}$$

SOLUTION

$$\left(\begin{array}{rrr|rrr} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{rrr|rrr} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrr|rrr} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right) \rightarrow \left(\begin{array}{rrr|rrr} 1 & 4 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right) \rightarrow \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right)$$

Thus

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

□

**EXAMPLE 8.** Solve the system

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 12 \\ -x_1 - 2x_2 \phantom{+ 3x_3} &= -12 \\ 2x_1 + 2x_2 + 3x_3 &= 8 \end{aligned}$$

The coefficient matrix of this system is the matrix $A$ of the last example. The solution to the system then is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 12 \\ -12 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -\frac{8}{3} \end{pmatrix}$$

□

## EXERCISES

**1.** Which of the following are elementary matrices? Classify each elementary matrix by type.

(a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$

**2.** Find the inverse of each of the matrices in Exercise 1. For each elementary matrix, verify that its inverse is an elementary matrix of the same type.

**3.** For each of the following pairs of matrices, find an elementary matrix $E$ such that $EA = B$.

(a) $A = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -4 & 2 \\ 5 & 3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{pmatrix}$

(c) $A = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 5 \end{pmatrix}$

**4.** For each of the following pairs of matrices, find an elementary matrix $E$ such that $AE = B$.

(a) $A = \begin{pmatrix} 4 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 1 & 4 \\ 4 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & 4 \\ 1 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix}$

(c) $A = \begin{pmatrix} 4 & -2 & 3 \\ -2 & 4 & 2 \\ 6 & 1 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -2 & 3 \\ -1 & 4 & 2 \\ 3 & 1 & -2 \end{pmatrix}$

**5.** Given

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{pmatrix}$$

(a) Find an elementary matrix $E$ such that $EA = B$.
(b) Find an elementary matrix $F$ such that $FB = C$.
(c) Is $C$ row equivalent to $A$? Explain.

**6.** Given

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{pmatrix}$$

(a) Find elementary matrices $E_1$, $E_2$, $E_3$ such that

$$E_3E_2E_1A = U$$

where $U$ is an upper triangular matrix.

(b) Determine the inverses of $E_1$, $E_2$, $E_3$ and set $L = E_1^{-1}E_2^{-1}E_3^{-1}$. What type of matrix is $L$? Verify that $A = LU$.

**7.** Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}$$

(a) Verify that

$$A^{-1} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix}$$

(b) Use $A^{-1}$ to solve $A\mathbf{x} = \mathbf{b}$ for the following choices of $\mathbf{b}$.

(i) $\mathbf{b} = (1, 1, 1)^T$
(ii) $\mathbf{b} = (1, 2, 3)^T$
(iii) $\mathbf{b} = (-2, 1, 0)^T$

**8.** Find the inverse of each of the following matrices.

(a) $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 6 \\ 3 & 8 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 0 \\ 9 & 3 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & 0 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$

(*) (g) $\begin{pmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{pmatrix}$

**9.** Given

$$A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

compute $A^{-1}$ and use it to:

(a) Find a $2 \times 2$ matrix $X$ such that $AX = B$.
(b) Find a $2 \times 2$ matrix $Y$ such that $YA = B$.

**10.** Given

$$A = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & -2 \\ -6 & 3 \end{pmatrix}$$

Solve each of the following matrix equations.

(*)

(a) $AX + B = C$
(b) $XA + B = C$
(c) $AX + B = X$
(d) $XA + C = X$

**11.** Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Show that if $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$, then

$$A^{-1} = \frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

**12.** Let $A$ be a nonsingular matrix. Show that $A^{-1}$ is also nonsingular and $(A^{-1})^{-1} = A$.

**13.** Prove that if $A$ is nonsingular, then $A^T$ is nonsingular and

$$(A^T)^{-1} = (A^{-1})^T$$

[***Hint:*** $(AB)^T = B^TA^T$.]

(*) **14.** Let $A$ be a nonsingular $n \times n$ matrix. Use mathematical induction to prove that $A^m$ is nonsingular and

$$(A^m)^{-1} = (A^{-1})^m$$

for $m = 1, 2, 3, \ldots$.

**15.** Is the transpose of an elementary matrix an elementary matrix of the same type? Is the product of two elementary matrices an elementary matrix?

**16.** Let $U$ and $R$ be $n \times n$ upper triangular matrices and set $T = UR$. Show that $T$ is also upper triangular and that $t_{jj} = u_{jj}r_{jj}$ for $j = 1, \ldots, n$.

**17.** Let $A$ and $B$ be $n \times n$ matrices and let $C = AB$. Prove that if $B$ is singular, then $C$ must be singular.

[***Hint:*** Use Theorem 1.4.3.]

**18.** Let $U$ be an $n \times n$ upper triangular matrix with nonzero diagonal entries.

(a) Explain why $U$ must be nonsingular.
(b) Explain why $U^{-1}$ must be upper triangular.

**19.** Let $A$ be a nonsingular $n \times n$ matrix and let $B$ be an $n \times r$ matrix. Show that the reduced row echelon form of $(A|B)$ is $(I|C)$, where $C = A^{-1}B$.

**20.** In general, matrix multiplication is not commutative (i.e., $AB \neq BA$). However, there are certain special cases where the commutative property does hold. Show that:

(a) If $D_1$ and $D_2$ are $n \times n$ diagonal matrices, then $D_1 D_2 = D_2 D_1$.
(b) If $A$ is an $n \times n$ matrix and

$$B = a_0 I + a_1 A + a_2 A^2 + \cdots + a_k A^k$$

where $a_0, a_1, \ldots, a_k$ are scalars, then $AB = BA$.

**21.** Show that if $A$ is a symmetric nonsingular matrix, then $A^{-1}$ is also symmetric.

**22.** Prove that if $A$ is row equivalent to $B$, then $B$ is row equivalent to $A$.

**23.** (a) Prove that if $A$ is row equivalent to $B$ and $B$ is row equivalent to $C$, then $A$ is row equivalent to $C$.
(b) Prove that any two nonsingular $n \times n$ matrices are row equivalent.

**24.** Prove that $B$ is row equivalent to $A$ if and only if there exists a nonsingular matrix $M$ such that $B = MA$.

**25.** Given a vector $\mathbf{x} \in R^{n+1}$, the $(n+1) \times (n+1)$ matrix $V$ defined by

$$v_{ij} = \begin{cases} 1 & \text{if } j = 1 \\ x_i^{j-1} & \text{for } j = 2, \ldots, n+1 \end{cases}$$

is called the Vandermonde matrix.

(a) Show that if

$$V\mathbf{c} = \mathbf{y}$$

and

$$p(x) = c_1 + c_2 x + \cdots + c_{n+1} x^n$$

then

$$p(x_i) = y_i, \quad i = 1, 2, \ldots, n+1$$

(b) Suppose that $x_1, x_2, \ldots, x_{n+1}$ are all distinct. Show that if $\mathbf{c}$ is a solution to $V\mathbf{x} = \mathbf{0}$, then the coefficients $c_1, c_2, \ldots, c_n$ must all be zero, and hence $V$ must be nonsingular.

## PARTITIONED MATRICES

Often it is useful to think of a matrix as being composed of a number of submatrices. A matrix $A$ can be partitioned into smaller matrices by drawing

# DETERMINANTS

With each square matrix it is possible to associate a real number called the determinant of the matrix. The value of this number will tell us whether or not the matrix is singular.

In Section 1 the definition of the determinant of a matrix is given. In Section 2 we study properties of determinants and derive an elimination method for evaluating determinants. The elimination method is generally the simplest method to use for evaluating the determinant of an $n \times n$ matrix when $n > 3$. In Section 3 we see how determinants can be applied to solving $n \times n$ linear systems and how they can be used to calculate the inverse of a matrix. An application involving cryptography is also presented in Section 3. Further applications of determinants are presented in Chapters 3 and 6.

## THE DETERMINANT OF A MATRIX

With each $n \times n$ matrix $A$ it is possible to associate a scalar, $\det(A)$, whose value will tell us whether or not the matrix is nonsingular. Before proceeding

**Case 1. $1 \times 1$ Matrices**

If $A = (a)$ is a $1 \times 1$ matrix, then $A$ will have a multiplicative inverse if and only if $a \neq 0$. Thus, if we define

$$\det(A) = a$$

then $A$ will be nonsingular if and only if $\det(A) \neq 0$.

**Case 2. $2 \times 2$ Matrices**

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

By Theorem 1.4.3, $A$ will be nonsingular if and only if it is row equivalent to $I$. Then if $a_{11} \neq 0$, we can test whether or not $A$ is row equivalent to $I$ by performing the following operations:

**1.** Multiply the second row of $A$ by $a_{11}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{11}a_{21} & a_{11}a_{22} \end{pmatrix}$$

**2.** Subtract $a_{21}$ times the first row from the new second row

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{11}a_{22} - a_{21}a_{12} \end{pmatrix}$$

Since $a_{11} \neq 0$, the resulting matrix will be row equivalent to $I$ if and only if

$$a_{11}a_{22} - a_{21}a_{12} \neq 0 \tag{1}$$

If $a_{11} = 0$, we can switch the two rows of $A$. The resulting matrix

$$\begin{pmatrix} a_{21} & a_{22} \\ 0 & a_{12} \end{pmatrix}$$

will be row equivalent to $I$ if and only if $a_{21}a_{12} \neq 0$. This requirement is equivalent to condition (1) when $a_{11} = 0$. Thus if $A$ is any $2 \times 2$ matrix and we define

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

then $A$ is nonsingular if and only if $\det(A) \neq 0$.

**Notation.** One can refer to the determinant of a specific matrix by enclosing the array between vertical lines. For example, if

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$$

then

$$\begin{vmatrix} 3 & 4 \end{vmatrix}$$

represents the determinant of $A$.

### Case 3. 3 × 3 Matrices

We can test whether or not a $3 \times 3$ matrix is nonsingular by performing row operations to see if the matrix is row equivalent to the identity matrix $I$. To carry out the elimination in the first column of an arbitrary $3 \times 3$ matrix $A$ let us first assume $a_{11} \neq 0$. The elimination can then be performed by subtracting $a_{21}/a_{11}$ times the first row from the second and $a_{31}/a_{11}$ times the first row from the third.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \dfrac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \dfrac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \\ 0 & \dfrac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}} & \dfrac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} \end{pmatrix}$$

The matrix on the right will be row equivalent to $I$ if and only if

$$a_{11} \begin{vmatrix} \dfrac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \dfrac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \\ \dfrac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}} & \dfrac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} \end{vmatrix} \neq 0$$

Although the algebra is somewhat messy, this condition can be simplified to

$$\begin{aligned} a_{11}a_{22}a_{33} &- a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} \\ &+ a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \neq 0 \end{aligned} \tag{2}$$

Thus if we define

$$\begin{aligned} \det(A) = {} & a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} \\ & + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \end{aligned} \tag{3}$$

then for the case $a_{11} \neq 0$ the matrix will be nonsingular if and only if $\det(A) \neq 0$.

What if $a_{11} = 0$? Consider the following possibilities:

(i) $a_{11} = 0, a_{21} \neq 0$
(ii) $a_{11} = a_{21} = 0, a_{31} \neq 0$
(iii) $a_{11} = a_{21} = a_{31} = 0$

In case (i), it is not difficult to show that $A$ is row equivalent to $I$ if and only if

$$-a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \neq 0$$

But this condition is the same as condition (2) with $a_{11} = 0$. The details of

In case (ii) it follows that

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is row equivalent to $I$ if and only if

$$a_{31}(a_{12}a_{23} - a_{22}a_{13}) \neq 0$$

Again this is a special case of condition (2) with $a_{11} = a_{21} = 0$.

Clearly, in case (iii) the matrix $A$ cannot be row equivalent to $I$ and hence must be singular. In this case if one sets $a_{11}$, $a_{21}$, and $a_{31}$ equal to 0 in formula (3), the result will be $\det(A) = 0$.

In general, then, formula (2) gives a necessary and sufficient condition for a $3 \times 3$ matrix $A$ to be nonsingular (regardless of the value of $a_{11}$).

We would now like to define the determinant of an $n \times n$ matrix. To see how to do this, note that the determinant of a $2 \times 2$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

can be defined in terms of the two $1 \times 1$ matrices

$$M_{11} = (a_{22}) \qquad \text{and} \qquad M_{12} = (a_{21})$$

The matrix $M_{11}$ is formed from $A$ by deleting its first row and first column and $M_{12}$ is formed from $A$ by deleting its first row and second column.

The determinant of $A$ can be expressed in the form

$$\text{(4)} \qquad \det(A) = a_{11}a_{22} - a_{12}a_{21} = a_{11}\det(M_{11}) - a_{12}\det(M_{12})$$

For a $3 \times 3$ matrix $A$ we can rewrite equation (3) in the form

$$\det(A) = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

For $j = 1, 2, 3$ let $M_{1j}$ denote the $2 \times 2$ matrix formed from $A$ by deleting its first row and $j$th column. The determinant of $A$ can then be represented in the form

$$\text{(5)} \qquad \det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + a_{13}\det(M_{13})$$

where

$$M_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}, \quad M_{12} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}, \quad M_{13} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

To see how to generalize (4) and (5) to the case $n > 3$, we introduce the following definition.

***Definition.*** Let $A = (a_{ij})$ be an $n \times n$ matrix. Let $M_{ij}$ be the $(n-1) \times (n-1)$

determinant of $M_{ij}$ is called the **minor** of $a_{ij}$. We define the **cofactor** $A_{ij}$ of $a_{ij}$ by

$$A_{ij} = (-1)^{i+j}\det(M_{ij})$$

In view of this definition, for a $2\times 2$ matrix $A$, we may rewrite equation (4) in the form

$$\text{(6)} \qquad \det(A) = a_{11}A_{11} + a_{12}A_{12} \qquad (n = 2)$$

Equation (6) is called the *cofactor expansion* of $\det(A)$ along the first row of $A$. Note that we could also write

$$\text{(7)} \qquad \det(A) = a_{21}(-a_{12}) + a_{22}a_{11} = a_{21}A_{21} + a_{22}A_{22}$$

Equation (7) expresses $\det(A)$ in terms of the entries of the second row of $A$ and their cofactors. Actually, there is no reason why we must expand along a row of the matrix; the determinant could just as well be represented by the cofactor expansion along one of the columns.

$$\begin{aligned}\det(A) &= a_{11}a_{22} + a_{21}(-a_{12})\\ &= a_{11}A_{11} + a_{21}A_{21} \qquad \text{(first column)}\\ \det(A) &= a_{12}(-a_{21}) + a_{22}a_{11}\\ &= a_{12}A_{12} + a_{22}A_{22} \qquad \text{(second column)}\end{aligned}$$

For a $3 \times 3$ matrix $A$, we have

$$\text{(8)} \qquad \det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Thus the determinant of a $3 \times 3$ matrix can be defined in terms of the elements in the first row of the matrix and their corresponding cofactors.

**EXAMPLE 1.** If

$$A = \begin{pmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{pmatrix}$$

then

$$\begin{aligned}\det(A) &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}\\ &= (-1)^2 a_{11}\det(M_{11}) + (-1)^3 a_{12}\det(M_{12})\\ &\quad + (-1)^4 a_{13}\det(M_{13})\\ &= 2\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} - 5\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 4\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix}\\ &= 2(6-8) - 5(18-10) + 4(12-5)\end{aligned}$$

As in the case of $2 \times 2$ matrices, the determinant of a $3 \times 3$ matrix can be represented as a cofactor expansion using any row or column. For example, equation (3) can be rewritten in the form

$$\begin{aligned}\det(A) &= a_{12}a_{31}a_{23} - a_{13}a_{31}a_{22} - a_{11}a_{32}a_{23} + a_{13}a_{21}a_{32} + a_{11}a_{22}a_{33} \\ &\quad - a_{12}a_{21}a_{33} \\ &= a_{31}(a_{12}a_{23} - a_{31}a_{22}) - a_{32}(a_{11}a_{23} - a_{13}a_{21}) \\ &\quad + a_{33}(a_{11}a_{22} - a_{12}a_{21}) \\ &= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}\end{aligned}$$

This is the cofactor expansion along the third row of $A$.

**EXAMPLE 2.** Let $A$ be the matrix in Example 1. The cofactor expansion of $\det(A)$ along the second column is given by

$$\begin{aligned}\det(A) &= -5\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} \\ &= -5(18 - 10) + 1(12 - 20) - 4(4 - 12) \\ &= -16\end{aligned}$$

□

The determinant of a $4 \times 4$ matrix can be defined in terms of a cofactor expansion along any row or column. To compute the value of the $4 \times 4$ determinant, one would have to evaluate four $3 \times 3$ determinants.

***Definition.*** The **determinant** of an $n \times n$ matrix $A$, denoted $\det(A)$, is a scalar associated with the matrix $A$ that is defined inductively as follows:

$$\det(A) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \text{if } n > 1 \end{cases}$$

where

$$A_{1j} = (-1)^{1+j}\det(M_{1j}) \qquad j = 1, \ldots, n$$

are the cofactors associated with the entries in the first row of $A$.

As we have seen, it is not necessary to limit ourselves to using the first row for the cofactor expansion. We state the following theorem without proof.

***Theorem 2.1.1.*** *If $A$ is an $n \times n$ matrix with $n \geq 2$, then* $\det(A)$ *can be expressed as a cofactor expansion using any row or column of $A$.*

$$\begin{aligned}\det(A) &= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \\ &= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}\end{aligned}$$

The cofactor expansion of a $4 \times 4$ determinant will involve four $3 \times 3$ determinants. One can often save work by expanding along the row or column that contains the most zeros. For example, to evaluate

$$\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{vmatrix}$$

one would expand down the first column. The first three terms will drop out, leaving

$$-2\begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix} = -2 \cdot 3 \cdot \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 12$$

The cofactor expansion can be used to establish some important results about determinants. These results are given in the following theorems.

**Theorem 2.1.2.** *If $A$ is an $n \times n$ matrix, then* $\det(A^T) = \det(A)$.

*Proof.* The proof is by induction on $n$. Clearly, the result holds if $n = 1$, since a $1 \times 1$ matrix is necessarily symmetric. Assume that the result holds for all $k \times k$ matrices and that $A$ is a $(k+1) \times (k+1)$ matrix. Expanding $\det(A)$ along the first row of $A$, we get

$$\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + - \cdots \pm a_{1,k+1}\det(M_{1,k+1})$$

Since the $M_{ij}$'s are all $k \times k$ matrices, it follows from the induction hypothesis that

$$(9) \quad \det(A) = a_{11}\det(M_{11}^T) - a_{12}\det(M_{12}^T) + - \cdots \pm a_{1,k+1}\det(M_{1,k+1}^T)$$

The right-hand side of (9) is just the expansion by minors of $\det(A^T)$ using the first column of $A^T$. Therefore,

$$\det(A^T) = \det(A)$$ □

**Theorem 2.1.3.** *If $A$ is an $n \times n$ triangular matrix, the determinant of $A$ equals the product of the diagonal elements of $A$.*

*Proof.* In view of Theorem 2.1.2, it suffices to prove the theorem for lower triangular matrices. The result follows easily using the cofactor expansion and induction on $n$. The details of this are left for the reader (see Exercise 8). □

**Theorem 2.1.4.** *Let $A$ be an $n \times n$ matrix.*

(i) *If $A$ has a row or column consisting entirely of zeros, then* $\det(A) = 0$.

(ii) *If $A$ has two identical rows or two identical columns, then* $\det(A) = 0$.

## PROPERTIES OF DETERMINANTS

In this section we consider the effects of row operations on the determinant of a matrix. Once these effects have been established, we will prove that a matrix $A$ is singular if and only if its determinant is zero and we will develop a method for evaluating determinants using row operations. Also, we will establish an important theorem about the determinant of the product of two matrices. We begin with the following lemma.

***Lemma 2.2.1.*** *Let $A$ be an $n \times n$ matrix. If $A_{jk}$ denotes the cofactor of $a_{jk}$ for $k = 1, \ldots, n$, then*

$$(1) \qquad a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

*Proof.* If $i = j$, (1) is just the cofactor expansion of $\det(A)$ along the $i$th row of $A$. To prove (1) in the case $i \neq j$, let $A^*$ be the matrix obtained by replacing the $j$th row of $A$ by the $i$th row of $A$.

$$A^* = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad j\text{th row}$$

Since two rows of $A^*$ are the same, its determinant must be zero. It follows from the cofactor expansion of $\det(A^*)$ along the $j$th row that

$$\begin{aligned} 0 = \det(A^*) &= a_{i1}A^*_{j1} + a_{i2}A^*_{j2} + \cdots + a_{in}A^*_{jn} \\ &= a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} \end{aligned}$$ □

Let us now consider the effects of each of the three row operations on the value of the determinant. We start with row operation II.

### ROW OPERATION II

*A row of $A$ is multiplied by a nonzero constant.*

Let $E$ denote the elementary matrix of type II formed from $I$ by multiplying along the $i$th row, then

$$\begin{aligned}\det(EA) &= \alpha a_{i1}A_{i1} + \alpha a_{i2}A_{i2} + \cdots + \alpha a_{in}A_{in} \\ &= \alpha(a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}) \\ &= \alpha \det(A)\end{aligned}$$

In particular,

$$\det(E) = \det(EI) = \alpha \det(I) = \alpha$$

and hence

$$\det(EA) = \alpha \det(A) = \det(E)\det(A)$$

## ROW OPERATION III

*A multiple of one row is added to another row.*

Let $E$ be the elementary matrix of type III formed from $I$ by adding $c$ times the $i$th row to the $j$th row. Since $E$ is triangular and its diagonal elements are all 1, it follows that $\det(E) = 1$. We will show that

$$\det(EA) = \det(A) = \det(E)\det(A)$$

If $\det(EA)$ is expanded by cofactors along the $j$th row, it follows from Lemma 2.2.1 that

$$\begin{aligned}\det(EA) &= (a_{j1} + ca_{i1})A_{j1} + (a_{j2} + ca_{i2})A_{j2} \\ &\quad + \cdots + (a_{jn} + ca_{in})A_{jn} \\ &= (a_{j1}A_{j1} + \cdots + a_{jn}A_{jn}) \\ &\quad + c(a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}) \\ &= \det(A)\end{aligned}$$

Thus

$$\det(EA) = \det(A) = \det(E)\det(A)$$

## ROW OPERATION I

*Two rows of A are interchanged.*

To see the effects of row operation I, we note that this operation can be accomplished using row operations II and III. We illustrate how this is done for $3 \times 3$ matrices.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Subtracting row 3 from row 2 yields

$$A^{(1)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - a_{31} & a_{22} - a_{32} & a_{23} - a_{33} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Next, the second row of $A^{(1)}$ is added to the third row:

$$A^{(2)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - a_{31} & a_{22} - a_{32} & a_{23} - a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Subtracting row 3 from row 2, we get

$$A^{(3)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -a_{31} & -a_{32} & -a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Since all of these matrices have been formed using only row operation III, it follows that

$$\det(A) = \det(A^{(1)}) = \det(A^{(2)}) = \det(A^{(3)})$$

Finally, if the second row of $A$ is multiplied through by $-1$, one obtains

$$A^{(4)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Since row operation II was used, it follows that

$$\det(A^{(4)}) = -1\det(A^{(3)}) = -\det(A)$$

$A^{(4)}$ is just the matrix obtained by interchanging the second and third rows of $A$.

This same argument can be applied to $n \times n$ matrices to show that whenever two rows are switched the sign of the determinant is changed. Thus if $A$ is $n \times n$ and $E_{ij}$ is the $n \times n$ elementary matrix formed by interchanging the $i$th and $j$th rows of $I$, then

$$\det(E_{ij}A) = -\det(A)$$

In particular,

$$\det(E_{ij}) = \det(E_{ij}I) = -\det(I) = -1$$

Thus for any elementary matrix $E$ of type I,

$$\det(EA) = -\det(A) = \det(E)\det(A)$$

In summation, if $E$ is an elementary matrix, then

$$\det(EA) = \det(E)\det(A)$$

where

$$(2) \qquad \det(E) = \begin{cases} -1 & \text{if } E \text{ is of type I} \\ \alpha \neq 0 & \text{if } E \text{ is of type II} \\ 1 & \text{if } E \text{ is of type III} \end{cases}$$

Similar results hold for column operations. Indeed, if $E$ is an elementary matrix, then

$$\begin{aligned} \det(AE) &= \det\big((AE)^T\big) = \det\big(E^T A^T\big) \\ &= \det\big(E^T\big)\det\big(A^T\big) = \det(E)\det(A) \end{aligned}$$

Thus the effects that row or column operations have on the value of the determinant can be summarized as follows:

**I.** Interchanging two rows (or columns) of a matrix changes the sign of the determinant.

**II.** Multiplying a single row or column of a matrix by a scalar has the effect of multiplying the value of the determinant by that scalar.

**III.** Adding a multiple of one row (or column) to another does not change the value of the determinant.

***Note.*** As a consequence of III, if one row (or column) of a matrix is a multiple of another, the determinant of the matrix must equal zero.

If follows from (2) that all elementary matrices have nonzero determinants. This observation can be used to prove the following theorem.

***Theorem 2.2.2.*** *An $n \times n$ matrix $A$ is singular if and only if*

$$\det(A) = 0$$

*Proof.* The matrix $A$ can be reduced to row echelon form with a finite number of row operations. Thus

$$U = E_k E_{k-1} \cdots E_1 A$$

where $U$ is in row echelon form and the $E_i$'s are all elementary matrices.

$$\begin{aligned} \det(U) &= \det(E_k E_{k-1} \cdots E_1 A) \\ &= \det(E_k)\det(E_{k-1}) \cdots \det(E_1)\det(A) \end{aligned}$$

Since the determinants of the $E_i$'s are all nonzero, it follows that $\det(A) = 0$ if and only if $\det(U) = 0$. If $A$ is singular, then $U$ has a row consisting entirely of zeros and hence $\det(U) = 0$. If $A$ is nonsingular, $U$ is triangular with 1's along the diagonal and hence $\det(U) = 1$. □

From the proof of Theorem 2.2.2 we can obtain a method for computing $\det(A)$. Reduce $A$ to row echelon form.

$$U = E_k E_{k-1} \cdots E_1 A$$

If the last row of $U$ consists entirely of zeros, $A$ is singular and $\det(A) = 0$. Otherwise, $A$ is nonsingular and

$$\det(A) = [\det(E_k)\det(E_{k-1}) \cdots \det(E_1)]^{-1}$$

Actually, if $A$ is nonsingular, it is simpler to reduce $A$ to triangular form. This can be done using only row operations I and III. Thus

$$T = E_m E_{m-1} \cdots E_1 A$$

and hence

$$\det(A) = \pm\det(T) = \pm t_{11} t_{22} \cdots t_{nn}$$

The sign will be positive if row operation I has been used an even number of times and negative otherwise.

**EXAMPLE 1.** Evaluate

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix}$$

SOLUTION

$$\begin{aligned} \begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} &= \begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{vmatrix} \\ &= (-1)\begin{vmatrix} 2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5 \end{vmatrix} \\ &= (-1)(2)(-6)(-5) \\ &= -60 \end{aligned}$$

□

We now have two methods for evaluating the determinant of an $n \times n$ matrix $A$. If $n > 3$ and $A$ has nonzero entries, elimination is the most efficient method in the sense that it involves less arithmetic operations. In Table 1 the number of arithmetic operations involved in each method is given for $n = 2, 3, 4, 5, 10$. It is not difficult to derive general formulas for the number of operations in each of the methods (see Exercises 16 and 17).

**TABLE 1**

| | Cofactors | | Elimination | |
|---|---|---|---|---|
| $n$ | Additions | Multiplications | Additions | Multiplications and Divisions |
| 2 | 1 | 2 | 1 | 3 |
| 3 | 5 | 9 | 5 | 10 |
| 4 | 23 | 40 | 14 | 23 |
| 5 | 119 | 205 | 30 | 45 |
| 10 | 3,628,799 | 6,235,300 | 285 | 339 |

We have seen that for any elementary matrix $E$,

$$\det(EA) = \det(E)\det(A) = \det(AE)$$

This is a special case of the following theorem.

***Theorem 2.2.3.*** *If $A$ and $B$ are $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B)$$

*Proof.* If $B$ is singular, it follows from Theorem 1.4.3 that $AB$ is also singular (see Exercise 15 of Chapter 1, Section 4), and therefore

$$\det(AB) = 0 = \det(A)\det(B)$$

If $B$ is nonsingular, $B$ can be written as a product of elementary matrices. We have already seen that the result holds for elementary matrices. Thus

$$\begin{aligned}\det(AB) &= \det(AE_kE_{k-1}\cdots E_1)\\ &= \det(A)\det(E_k)\det(E_{k-1})\cdots\det(E_1)\\ &= \det(A)\det(E_kE_{k-1}\cdots E_1)\\ &= \det(A)\det(B)\end{aligned}$$

□

If $A$ is singular, the computed value of $\det(A)$ using exact arithmetic must be 0. However, this result is unlikely if the computations are done by computer. Since computers use a finite number system, roundoff errors are usually unavoidable. Consequently, it is more likely that the computed value of $\det(A)$ will only be near 0. Because of roundoff errors, it is virtually impossible to determine computationally whether or not a matrix is exactly singular. In computer applications it is often more meaningful to ask whether a matrix is "close" to being singular. In general, the value of $\det(A)$ is not a good indicator of nearness to singularity. In Chapter 7 we will discuss how to determine whether or not a matrix is close to being singular.

## EXERCISES

**1.** Evaluate each of the following determinants by inspection.

(a) $\begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

(*) **2.** Let

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{pmatrix}$$

(a) Use the elimination method to evaluate $\det(A)$.
(b) Use the value of $\det(A)$ to evaluate

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 4 & 4 \\ 2 & 3 & -1 & -2 \end{vmatrix}$$

**3.** For each of the following, compute the determinant and state whether the matrix is singular or nonsingular.

(a) $\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 5 \\ 2 & 1 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{pmatrix}$

**4.** Find all possible choices of $c$ that would make the following matrix singular.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{pmatrix}$$

**5.** Let $A$ be an $n \times n$ matrix and $\alpha$ a scalar. Show that

$$\det(\alpha A) = \alpha^n \det(A)$$

**6.** Let $A$ be a nonsingular matrix. Show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

**7.** Let $A$ and $B$ be $3 \times 3$ matrices with $\det(A) = 4$ and $\det(B) = 5$. Find the value of:
(a) $\det(AB)$ (b) $\det(3A)$ (c) $\det(2AB)$ (d) $\det(A^{-1}B)$

**8.** Let $E_1$, $E_2$, $E_3$ be $3 \times 3$ elementary matrices of types I, II, and III, respectively, and let $A$ be a $3 \times 3$ matrix with $\det(A) = 6$. Assume, additionally, that $E_2$ was formed from $I$ by multiplying its second row by 3. Find the values of each of the following.
(a) $\det(E_1 A)$ (b) $\det(E_2 A)$ (c) $\det(E_3 A)$
(d) $\det(A E_1)$ (e) $\det(E_1^2)$ (f) $\det(E_1 E_2 E_3)$

**9.** Let $A$ and $B$ be row equivalent matrices and suppose that $B$ can be obtained from $A$ using only row operations I and III. How do the values of $\det(A)$ and $\det(B)$ compare? How will the values compare if $B$ can be obtained from $A$ using only row operation III? Explain your answers.

**10.** Consider the $3 \times 3$ Vandermonde matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$$

(a) Show that $\det(V) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$.
(b) What conditions must the scalars $x_1$, $x_2$, $x_3$ satisfy in order for $V$ to be nonsingular?

**11.** Suppose that a $3 \times 3$ matrix $A$ factors into a product

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$