THE PP CONJECTURE IN THE THEORY OF SPACES OF ORDERINGS

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Paweł Gładki

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University of Saskatchewan

Saskatoon, Saskatchewan, Canada

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Abstract

The notion of spaces of orderings was introduced by Murray Marshall in the 1970's and provides an abstract framework for studying orderings on fields and the reduced theory of quadratic forms over fields. The structure of a space of orderings (X, G) is completely determined by the group structure of G and the quaternary relation $(a_1, a_2) \cong (a_3, a_4)$ on G – the groups with additional structure arising in this way are called reduced special groups. The theory of reduced special groups, in turn, can be conveniently axiomatized in the first order language L_{SG}. Numerous important notions in this theory, such as isometry, isotropy, or being an element of a value set of a form, make an extensive use of, so called, positive primitive formulae in the language L_{SG}. Therefore, the following question, which can be viewed as a type of very general and highly abstract local-global principle, is of great importance:

Is it true that if a positive primitive formula holds in every finite subspace of a space of orderings, then it also holds in the whole space?

This problem is now known as the pp conjecture. The answer to this question is affirmative in many cases, although it has always seemed unlikely that the conjecture has a positive solution in general. In this thesis, we discuss, discovered by us, first counterexamples for which the pp conjecture fails. Namely, we classify spaces of orderings of function fields of rational conics with respect to the pp conjecture, and show for which of such spaces the conjecture fails, and then we disprove the pp conjecture for the space of orderings of the field $\mathbb{R}(x, y)$. Some other examples, which can be easily obtained from the developed theory, are also given. In addition, we provide a refinement of the result previously obtained by Vincent Astier and Markus Tressl, which shows that a pp formula fails on a finite subspace of a space of orderings, if and only if a certain family of formulae is verified.

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Contents

Introduction

The concept of an ordered field goes back to David Hilbert and his 1899 work on the foundations of geometry. In §13 of [**Hil99**] he introduced the notion of, what he called, "complex number systems", and listed 16 axioms of ordered fields, as well as the Archimedean axiom. This axiomatization was later used in §28, where he showed that the "algebra of segments", constructed to study plane geometry, was indeed an ordered field, and also in §29, where the first example of a non-Archimedean geometry was built. The significance of this discovery was widely appreciated; Henri Poincaré, well-known for his hostility towards the formalist viewpoint, wrote in his review of Hilbert's book:

> This notion may seem artificial and puerile; and it is needless to point out how disastrous it would be in teaching and how hurtful in mental development; how deadening it would be for investigators, whose originality it would nip in the bud. But, as used by Professor Hilbert, it explains and justifies itself, if one remembers the end pursued [**Poi02**].

However, it was not until the series of papers by Emil Artin and Otto Schreier published in 1926 and 1927, that the systematic development of the theme of orderings of fields started. The fifth issue of *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* contained three fundamental works [**Art27**], [**ArtSch27-1**] and [**ArtSch27-2**], where it was shown that fields admitting orderings are those in which -1 is not a sum of squares, that an element of a field is a sum of squares if and only if it is positive with respect to every order in that field, and where the notion of real closed fields was first introduced, along with the proof that every real closed field admits the unique order, and every ordered field has a unique, up to an isomorphism, real closed algebraic extension, called the real closure, whose ordering induces the ordering of the underlying field; last but not least, all those considerations led to the solution of the celebrated Hilbert's 17th Problem. The importance of this theory was fully acknowledged in 1931 by Bartel van der Waerden, when he devoted the whole Chapter 11 of his book *Moderne Algebra* [**Wae31**] to real fields, which therefrom became a part of every standard algebra textbook.

Artin's solution of Hilbert's problem related for the first time the theory of ordered fields with the real algebraic geometry; his proof used, along with the newly developed Artin-Schreier theory, a specialization argument and Sturm's theorem on counting the real zeros of polynomials. This relationship became even more evident in Serge Lang's proof of the same theorem [**Lan65**], where he used his Homomorphism Theorem and replaced the specialization argument with the use of real places [**Lan53**]. These techniques later led to the Real Nullstellensatz by Didier Dubois [**Dub70**] and Jean-Jacques Risler [**Ris70**]. On the other hand, Alfred Tarski discovered his famous Tarski Transfer Principle (first announced without proof in [**Tar31**], later published in [**Tar51**]), whilst Abraham Robinson proved the model completeness of the elementary theory of real closed fields [**Rob56**]; these results greatly contributed to the field of model theory. The Artin-Schreier theory has been also applied to the algebraic theory of quadratic forms; in our work we shall concentrate on this application.

The relationship between orderings and quadratic forms traces back to works by James Joseph Sylvester [**Syl52**] and his notion of a signature, which was later revitalized by Albrecht Pfister in the proof of his celebrated Local-Global Principle [**Pfi66**]. For a given ordering P and a quadratic form ϕ over a formally real field F, he defined the signature $\operatorname{sgn}_P(\phi)$, just as Sylvester did for the field $\mathbb{R}$. If X_F denotes the set of all orderings of the field F, this gave a rise to a "total signature" of the form ϕ, that is the function $\operatorname{Sgn}_\phi : X_F \to \mathbb{Z}$ defined by the following formula:

$$\operatorname{Sgn}_\phi(P) = \operatorname{sgn}_P(\phi).$$

Assigning to every form ϕ its total signature Sgn_ϕ yielded a well-defined homomorphism from the Witt ring $W(F)$ of the field F to the ring $C(X_F, \mathbb{Z})$ of continuous functions defined over the set X_F (with suitably chosen topology) with values in $\mathbb{Z}$. Pfister's main result stated that the kernel of this homomorphism was precisely the torsion ideal $W_t(F)$ of $W(F)$. Therefore, the study of the reduced Witt ring $W(F)/W_t(F)$ is essentially tied to the study of the space of orders of the field F.

This theory was soon generalized to subspaces of spaces of orderings, which, in turn, first appeared in works of Jean-Pierre Serre [**Ser49**], where he attempted to – according to his own words – "try to catch the orderings" and introduced the notion of preorderings. If T is a fixed preordering of a formally real field F, let X_T denote the set of orders of F extending T (we hope that the reader shall not be confused with the similarity of the notation X_F and X_T, which seems to be widely accepted in the literature). In papers by Bröcker [**Brö74**], Becker and Köpping [**BecKöp77**], Scharlau [**Sch69**], Knebusch, Rosenberg and Ware [**KneRosWar73**], and Marshall [**Mar77**] results similar to the mentioned above were proven, with the space of orderings X_F replaced with X_T (and, even more generally, with spaces of orders over commutative rings instead of just over fields), which paved the way to the reduced theory of quadratic forms. The systematic exposition of the latter one was given, for example, in [**Lam81**]. It turned out, that it was possible to construct a very elegant theory in which reduced Witt rings were constructed in a natural way, analogous to the classical case.

The observation that the space X_T of orderings extending a given preordering T of a field F may be viewed as a subset of the character group $\chi(G)$ of the group $G = (F \setminus \{0\})/(T \setminus \{0\})$ of generalized square classes of F with respect to T, led to the development of an extensive axiomatic theory of abstract ordering spaces, where arbitrary elementary Abelian 2-groups are considered. This theory was constructed by Murray Marshall in a series of papers [**Mar76**], [**Mar79-1**], [**Mar79-2**], [**Mar80-1**], [**Mar80-2**] and [**Mar80-3**].

This was not the only attempt to develop the algebraic theory of quadratic forms on an axiomatic basis. Most notable among other works on that theme is the one by Murray Marshall, where the quaternionic structures are introduced [**Mar80-3**], and the one by Mieczysław Kula, Lucyna Szczepanik and Kazimierz Szymiczek [**KulSzcSzy88**], where the quaternionic schemes are studied. In the beginning of the 1990s the notion of special groups was introduced by Max Dickmann [**Dic93**]; there exists an algebraic-topological duality between the category of reduced special groups and that of abstract spaces of orderings, which was studied in detail, for example, by Arileide Lira De Lima in [**Lim93-1**] and in her thesis [**Lim93-2**]. The monograph [**DicMir00**] provides a detailed discussion of these topics. Due to the mentioned duality, the language of reduced special groups is a convenient and frequently used tool to work with spaces of orderings.

The notion of positive primitive (abbreviated pp) formulae appeared in the study of pure embeddings in the theory of reduced special groups [**DicMir00**]. This terminology was borrowed from the theory of modules [**Ho93**, p. 56], where pure embeddings are rather natural objects, and some examples arise in the study of fields. In general, a pp formula is a formula of the form

$$P(a_1,\dots,a_k) = \exists t_1 \dots \exists t_n [\theta_1(t_1,\dots,t_n,a_1,\dots,a_k) \wedge \dots \wedge \theta_m(t_1,\dots,t_n,a_1,\dots,a_k)]$$

where $\theta_1,\dots,\theta_m$ are atomic formulae and $a_1,\dots,a_k$ are some parameters. In order to shorten this rather lengthy and inconvenient notation, we shall simply write

$$P(\underline{a}) = \exists \underline{t} \bigwedge_{j=1}^{m} \theta_j(\underline{t},\underline{a}),$$

where $\underline{a} = (a_1,\dots,a_k)$ and $\underline{t} = (t_1,\dots,t_n)$. In the language of reduced special groups numerous important properties of quadratic forms over spaces of orderings can be expressed as pp formulae. In particular, "two forms are isometric", "an element is represented by a form", or "a form is isotropic" are all examples of pp formulae [**Mar96**]. The following question, which can be viewed as a type of very general and highly abstract local-global principle, and which is now known as the pp conjecture, was posed by M. Marshall in [**Mar02**]:

Is it true that if a pp formula holds in every finite subspace of a space of orderings, then it also holds in the whole space?

The answer to this question is affirmative for all the examples of pp formulae mentioned above; for the formula "two forms are isometric" this fact is a trivial observation, for the formula "an element is represented by a form" this is a deep result first proven by Becker and Bröcker in [**BecBrö78**], and later proven in the context of spaces of orderings by Marshall in [**Mar80-1**]. Another example of an important pp formula for which the pp conjecture is true is provided by the Extended Isotropy Theorem, discussed for the first time by Marshall in [**Mar84**]. In a recent paper [**Mar06**] a still larger class of pp formulae, called product-free and one-related, is introduced and it is shown that, for every such formula and for any space of orderings having finite stability index, the answer to the pp conjecture is "yes". It has also been proven, that the class of spaces for which the conjecture is true for every pp formula contains spaces of orderings of finite chain length, spaces of orderings of stability index 1 (which includes spaces of orderings of curves over real closed fields), is closed under direct sum and group extension (see [**Mar02**]), and under the operation of taking subspaces (which is a consequence of results by Vincent Astier and Markus Tressl presented in [**AstTre05**]).

It has always seemed unlikely that the conjecture has an affirmative solution in general, though no examples had been known until quite recently. A positive answer would automatically imply a positive answer to a question of representation modulo 2^n posed by Michel Coste in 1999; this problem is discussed in Isabelle Bonnard's work [**Bon00**]. Another consequence would be the complete solution of Lam's Open Problem B, formulated by Lam in [**Lam77**], and recently solved in the field case by Dickmann and Miraglia [**DicMir03**]. If the pp conjecture was true, that would also provide a positive answer to the separating depth problem stated in [**Mar94**] which, in turn, relates to Bröcker question about the relationship between the stability index and the t-invariant (see [**Brö84**]). All mentioned relationships are studied in details in [**Mar02**].

In view of the above remarks, it has been always desired to find some counterexamples to the pp conjecture. Due to the mentioned results, such counterexamples cannot be found among spaces of orderings of stability index 1. As of spaces of higher stability index, both the space of orderings of the field $\mathbb{Q}(x)$, spaces of orderings of function fields of rational conic sections, and the space of orderings of the field $\mathbb{R}(x, y)$ have stability index 2. For the first one, the conjecture holds true, which was shown by Dickmann, Marshall and Miraglia in

[**DicMarMir05**]; in this thesis we concentrate on the remaining two cases – following our work in [**GłaMar-1**] and [**GłaMar-2**], we classify spaces of orderings of conics with respect to whether the pp conjecture holds true or not, and also give a negative solution to the conjecture for the space of orderings of $\mathbb{R}(x, y)$. We also discuss some general properties of pp formulae, such as behavior of the pp conjecture in subspaces of spaces of orderings, or constructions of families of formulae testing a given pp formula on finite subspaces of some space of orderings; these topics have not been covered in our previous papers.

In the first chapter we introduce some basic notions in the theory of spaces of orderings. These are classical results and the reader who is already familiar with the notion of spaces of orderings and special groups may wish to skip this part of our work. There are essentially no proofs included in this chapter – only a few lemmas and examples at the beginning are explained in some detail, with the intention of providing as gentle and painless introduction to the subject as possible. We show how the concept of ordering relations in fields can be generalized to the notion of spaces of orderings, and we give some examples of such spaces, with emphasis on the relationship between the discussed theory and the theory of valuations. Next, we define quadratic forms in spaces of orderings, and we explain how the well known Sylvester's criterion for isometry of quadratic forms over the field $\mathbb{R}$ leads to the definition of isometry of quadratic forms in spaces of orderings – similarly, we define value sets of quadratic forms in spaces of orderings, and show how this definition relates to what we understand as a "classical" definition of a value set. We then proceed to investigate some of the basic properties of isometry and value sets, and recall the notions of subspaces of spaces of orderings, of the Harrison topology, of fans, generating sets and dual bases, of the stability index and the strong approximation property, of group extensions, direct sums and connected components of spaces of orderings, and, finally, of the chain length. We quote major theorems that shall be frequently used in the course of our work, such as the Structure Theorem or the Isotropy Theorem, and we conclude this chapter with brief introduction to the theory of reduced special groups.

In Chapter 2 we introduce pp formulae and formally state the pp conjecture. We then investigate some basic properties of pp formulae; in Section 1 we quote some old results, and in Section 2 we prove a theorem explaining how the pp conjecture behaves in subspaces (Theorem 2.2.1). This result is used to show how the pp conjecture is preserved under direct sums and group extensions. In Section 3 we define product free and one-related pp formulae, and quote the result stating that the pp conjecture holds for this type of formulae in spaces of orderings of finite stability index (Theorem 2.3.1). Finally, in Section 4 we investigate products of value sets of quadratic forms, which lead us to some examples of pp formulae which are not product free or one-related – proofs of few well known lemmas which will be used later in the work are also given for completeness.

Chapter 3 is the extended version of our paper [**GłaMar-1**]; after introducing the necessary tools in Section 1, in Sections 2, 3 and 4 we classify spaces of orderings of function fields of rational conic sections with respect to the pp conjecture. We recall that every rational conic is affine isomorphic to a parabola, an ellipse or a hyperbola, or to two parallel lines. Function fields of irreducible rational parabolas are isomorphic to $\mathbb{Q}(x)$, and in this case the question of validity of the pp conjecture is settled by the result in [**DicMarMir05**] mentioned earlier. Irreducible conics with rational points are either two parallel lines, or hyperbolas, or ellipses. Irreducible "degenerate" ellipses or hyperbolas of the form $ax^2 + by^2 = 0$, $a, b \neq 0$ are birationally equivalent to two parallel lines without a rational point. Moreover, function fields of irreducible rational hyperbolas or ellipses with rational points are isomorphic to $\mathbb{Q}(x)$. Thus we are down to considering irreducible hyperbolas or ellipses, and irreducible parallel lines without rational points. For each of those two types of curves we construct pp formulae which hold true in every finite subspace of the space of orderings of the function field of the given curve, but fail in the whole space, and thus we provide counterexamples to the pp conjecture. The key ingredient of the proof is the fact that the coordinate ring of an ellipse, or a hyperbola, or two parallel lines without rational points, is a principle ideal domain, which allows us to describe all valuations of the function field of such a curve; this result is an easy observation in the case of two parallel lines, while for an ellipse or a hyperbola a finer argument is required – the respective theorem has been known at least since the 1960's (see [**Sam61**]), but an elegant and elementary proof is given here for completeness. We conclude this chapter with Section 5, where we show how the results obtained before can be used to disprove the pp conjecture for the field $\mathbb{Q}(x_1, \ldots, x_n)$, $n \geq 2$.

Chapter 4 is concerned with the third of the above mentioned examples of spaces of orderings of stability index 2, and in Section 1 we show how the pp conjecture can fail for the space of orderings of the field $\mathbb{R}(x, y)$. Due to rather complicated valuations of that field here new, "valuation theory free" methods are developed and used. This chapter basically covers the material contained in our paper [**GłaMar-2**]. In Section 2 we disprove the pp conjecture for spaces of orderings of fields $\mathbb{R}(x_1, \ldots, x_n)$, $n \geq 2$, and we state some open questions. In particular, we ask for validity of the pp conjecture in spaces of orderings of formally real finitely generated extensions of $\mathbb{R}$ of transcendence degree at least two, or in spaces of orderings of power series fields. We also discuss the case of the field $\mathbb{R}$ replaced with an arbitrary real closed field R.

We conclude our work with Chapter 5, where we present some refinements of the results previously obtained by Astier and Tressl in [**AstTre05**]. The main result proven in [**AstTre05**] is a theorem, which shows that a pp formula fails on a finite subspace of a space of orderings if and only if a certain family of formulae is verified. We strengthen this result by constructing another family of formulae with the same property, whose elements are given explicitly. In the final section of our work we show how these considerations can be applied to some problems concerning the pp conjecture, such as determining whether the conjecture holds true on subspaces of a given space of orderings. We also give another proof of the theorem stating that the pp conjecture is preserved with respect to subspaces of a space of orderings.

List of notation

We will use standard notation for sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. For a subset A of a field K we will write A^* for the set $A \setminus \{0\}$, and for a domain D we shall denote by (D) its field of fractions. A list of some more special symbols used in the text is given below.

CHAPTER 1

Preliminaries

The theory of spaces of orderings was developed by Murray Marshall in a series of papers [**Mar76**], [**Mar79-1**], [**Mar79-2**], [**Mar80-1**], [**Mar80-2**] and [**Mar80-3**], and this chapter briefly outlines main definitions and theorems that will be later used in our work. For the reader's convenience all references here point to the monograph [**Mar96**], which provides a systematic treatment of the theory. The Baer-Krull correspondence is explained in a manner borrowed from [**Mar00**], and the reader more interested in the subject should probably refer either to that book, or to [**Pre84**]. The last section, where the theory of special groups is discussed, is based on [**DicMir03**]. A good reference for the reduced theory of quadratic forms and interconnections between valuations, orderings and quadratic forms is [**Lam81**].

1.1. Orderings and preorderings of fields

Let K be a field. Some of the following results hold true in every field K, but we will assume for simplicity that $\mathrm{char} K \neq 2$. Denote by K^2 the set of squares of elements of K and by ΣK^2 the set of (finite) sums of squares of K. A **preordering** of K is a subset $T \subset K$ closed under addition and multiplication, and containing the set K^2, i.e., $T + T \subset T$, $TT \subset T$, $K^2 \subset T$. For example, the set of all nonnegative reals is a preordering of $\mathbb{R}$, ΣK^2 is a preordering of every field K – moreover, ΣK^2 is the unique smallest preordering of K – the set $\Sigma K^2[S]$ of all finite sums of elements of the form $\sigma^2 g_1 \dots g_s$, for $\sigma \in K$, $g_1, \dots, g_s \in S$, $s \in \mathbb{N}$, where $S \subset K$ is some subset, is the preordering of K generated by the set S. A preordering T of K is called a **proper preordering** if $T \subsetneq K$; we shall mainly deal with proper preorderings and exclude cases when, e.g. K is algebraically closed and $\Sigma K^2 = K$.

By Zorn's Lemma, every proper preordering can be extended to an **ordering**, that is, a subset $P \subset K$ such that $P + P \subset P$, $PP \subset P$, $P \cup -P = K$, and $P \cap -P = \{0\}$, where

$-P = \{a \in K : -a \in P\}$. Actually, one has a slightly finer result ([**Mar96**, Theorem 1.1.1]):

1.1.1. LEMMA. *Let $T \subset K$ be a preordering,* char$K \neq 2$*, let $a \in K \setminus T$. Then there is an ordering $P \subset K$, such that $T \subset P$ and $a \notin P$.*

PROOF. One easily verifies that $T' = \{s - at : s, t \in T\}$ is also a preordering in K. Observe that $-1 \notin T'$: for if $-1 = s - at \in T'$ for some choice of $s, t \in T$, then $at = 1 + s$. If $1 + s = 0$, then $-1 = s \in T$, so $a = (\frac{a+1}{2})^2 - (\frac{a-1}{2})^2 = (\frac{a+1}{2})^2 + (-1)(\frac{a-1}{2})^2 \in T$ – a contradiction. If $1 + s \neq 0$, then $t \neq 0$, so $a = \frac{1+s}{t} = (\frac{1}{t})^2(t)(1+s) \in T$ – a contradiction.

By Zorn's Lemma, there is a preordering P maximal subject to the conditions $P \supset T'$ and $-1 \neq P$. We shall see that P is also an ordering: for if $b \in K$ and $b \notin P$ then, as before, $P' = \{p - br : p, r \in P\}$ is a preordering, $-1 \notin P'$ and $P \subset P'$, so, by maximality of P, $P = P'$. In particular, $-b \in P$, so that $P \cup -P = K$. Moreover, if $c \in P \cap -P$ and $c \neq 0$, then $-1 = (\frac{1}{c})^2(c)(-c) \in P$ – a contradiction. Therefore $P \cap -P = \{0\}$.

Finally, $a \neq 0$ (0 is a square) and $-a \in P$, so $a \notin P$. $\square$

If P is an ordering of K and $a, b \in K$, we write $a \leq_P b$ to indicate that $b - a \in P$. $\leq_P$ is a relation of total order on K which is also compatible with addition and multiplication (that is, if $a, b, c \in K$ and $a \leq_P b$, then $a + c \leq_P b + c$ and if, moreover, $0 \leq_P c$ then $ac \leq_P bc$). Thus $P = \{a \in K : 0 \leq_P a\}$ and P symbolizes the set of "nonnegative" elements. A field equipped with an ordering shall be called a **formally real** field.

Of course every ordering is also a preordering, but the converse is not true – although $\Sigma\mathbb{R}^2$ and $\Sigma\mathbb{Q}^2$ are orderings (obviously $\frac{m}{n} = \left(\frac{1}{n}\right)^2(mn) = \underbrace{\left(\frac{1}{n}\right)^2 + \ldots + \left(\frac{1}{n}\right)^2}_{mn}$; actually, by the Lagrange's Four Squares Theorem, a much stronger result is valid), the set $\Sigma\mathbb{R}(X)^2$ is a preordering, but not an ordering: a polynomial attaining both positive and negative values clearly can not be $\pm$ a sum of squares of rational functions.

1.2. Orderings and valuations

Let K be a field, char$K \neq 2$. Let X_K denote the set of all orderings of K. We shall describe the structure of the set X_K in more detail. An ordering P of K is called **Archimedean** if, for each $a \in K$, there exists an integer $n \geq 1$ such that $n - a, n + a \in P$ (or, equivalently, just $n + a \in P$). Recall that a surjective map $v : K \to G \cup \{\infty\}$, where G is an ordered additive Abelian group (that is, an Abelian group endowed with a total ordering compatible with addition) and ∞ is larger than any element of G, is called a **valuation** of K if

(1) $v(a) = \infty$ if and only if $a = 0$,

(2) $\forall a, b \in K \setminus \{0\}(v(ab) = v(a) + v(b))$,

(3) $\forall a, b \in K \setminus \{0\}(v(a + b) \geq \min\{v(a), v(b)\})$.

A valuation v such that $v(a) = 0$ if and only if $a \neq 0$ is called the **trivial valuation.** We introduce the usual notation:

$$A_v = \{a \in K : v(a) \geq 0\}, \quad M_v = \{a \in K : v(a) > 0\}, \quad U_v = A_v \setminus M_v.$$

A_v is a **valuation ring** of K (that is a subring such that, for all $a \in K$, if $a \notin A_v$ then $a^{-1} \in A_v$), and M_v is its unique maximal ideal. In particular, $K_v = A_v/M_v$ is a field called the **residue field** of v.

An ordering P of K will be called **compatible** with a valuation v of K if

$$\forall a, b \in K(0 <_P a \leq_P b \Rightarrow v(a) \geq v(b)).$$

We will denote by X_v the set of all orderings of K compatible with v. The set X_K is the union of the set of all Archimedean orderings of K and the sets X_v, where v is a nontrivial valuation of K whose residue field K_v is formally real (see [**Pre84**, Theorem 7.14]).

Archimedean orderings on K arise from embeddings $K \hookrightarrow \mathbb{R}$ (compare [**Pre84**, Theorem 1.24]) by taking the counter-images of the nonnegative reals via such embeddings. Let, for example, K be a number field, i.e. $K \cong \mathbb{Q}[x]/(p)$, where p is an irreducible polynomial. Then every ordering on K is Archimedean, and the number of embeddings of K into $\mathbb{R}$ is equal to the number of real roots of p. For example, x^2-2 has two real roots, so $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}[x]/(x^2-2)$ has two orderings, one making $\sqrt{2}$ positive, and the other one making $\sqrt{2}$ negative.

Orderings compatible to a valuation v of K are described by the Baer-Krull correspondence ([**Pre84**, Lemma 7.5, Lemma 7.7]), which we shall now briefly outline. Let $v : K \to G \cup \{\infty\}$

be a valuation with formally real residue field K_v. Let Q be an ordering of K_v and define

$$U_v^+ = \{a \in U_v : a + M_v \in Q\}.$$

Denote by $(K^*)^2$ the set $\{a^2 : a \in K^*\}$. Observe that $-1 \notin U_v^+(K^*)^2$, for if $-1 = ab^2$, $a \in U_v^+$, $b \in K^*$, then, by comparing values, $v(b) = 0$, and $b + M_v \in K_v$ is well defined. Further, $a + M_v \in Q$ and, consequently, $-1 + M_v = (a + M_v)(b + M_v)^2 \in Q$ – a contradiction. By Zorn's Lemma, there is a subgroup P^* of K^* containing $U_v^+(K^*)^2$ and maximal subject to $-1 \notin P^*$. $P = P^* \cup \{0\}$ is an ordering of K compatible with v ([**Mar96**, Theorem 1.3.1]).

Any subgroup P^* of K^* containing $U_v^+(K^*)^2$ and maximal subject to $-1 \notin P^*$ has index 2 in K^2, and thus the set of such subgroups corresponds in a natural way to **characters** of the group K^* into the group $\{-1, 1\}$ (that is group homomorphisms $\chi : K^* \to \{-1, 1\}$; we shall refer to the characters into $\{-1, 1\}$ simply as to the characters) which satisfy:

$$(1)\ \chi(-1) = -1, \text{ and } (2)\ U_v^+(K^*)^2 \subset \ker \chi$$

. Indeed, if P^* is such a subgroup, then the function $\chi : K^* \to \{-1, 1\}$ given by

$$\chi(a) = \begin{cases} 1, & \text{if } a \in P^*, \\ -1, & \text{if } a \notin P^* \end{cases}$$

is a character satisfying (1) and (2). Conversely, for a character $\chi : K^* \to \{-1, 1\}$ satisfying (1) and (2), $P^* = \ker \chi$ is a subgroup of K^* of index 2 such that $-1 \notin P^*$ and $U_v^+(K^*)^2 \subset P^*$.

Fix a character $\chi_0 : K^* \to \{-1, 1\}$ satisfying (1) and (2). Since $U_v = U_v^+ \cup -U_v^-$, $\chi : K^* \to \{-1, 1\}$ is a character satisfying (1) and (2) if and only if $\chi = \chi_0 \rho$, where $\rho : K^* \to \{-1, 1\}$ is a character trivial on $U_v(K^*)^2$. Thus the set of characters of K^* satisfying (1) and (2) corresponds to the set of characters of K^* trivial on $U_v(K^*)^2$.

The set of such characters is a group isomorphic to the character group of $K^*/U_v(K^*)^2$. Finally, $K^*/U_v(K^*)^2 \cong G/2G$ via the mapping $aU_v(K^*)^2 \mapsto v(a) + 2G$. To sum up, we have:

1.2.1. Lemma. *For a fixed valuation $v : K \to G \cup \{\infty\}$ with formally real residue field, the set X_v is in a one-to-one correspondence to the set $X_{K_v} \times \chi(G/2G)$, $\chi(G/2G)$ denoting the character group of the group $G/2G$.*

Therefore, we have a clear description of the set X_K as long as the valuations of the field K are well understood. Take, for example, $K = \mathbb{Q}(x)$ and let $p \in \mathbb{Q}[x]$ be an irreducible monic polynomial. The polynomial p gives a rise to a valuation $v : \mathbb{Q}(x) \to \mathbb{Z} \cup \{\infty\}$, which acts on polynomials as follows: if $f = p^k g$ and $p \nmid g$, $f, g \in \mathbb{Q}[x]$, then $v(f) = k$. The residue field $\mathbb{Q}(x)_v$ is isomorphic to $\mathbb{Q}[x]/(p)$, and the number of orderings of $\mathbb{Q}(x)_v$ is equal to the number of real roots of p. Say ξ is one of these roots – then the evaluation map $\mathbb{Q}[x] \ni f \mapsto f(\xi) \in \mathbb{R}$ gives a rise to an ordering Q of $\mathbb{Q}(x)_v$, and $g + M_v \in Q$ if and only if $g(\xi) > 0$, $g \in \mathbb{Q}[x]$, $p \nmid g$. Next, there are two characters of the group $\mathbb{Z}/2\mathbb{Z} \cong \{-1, 1\}$, one which maps -1 to -1, and the other one mapping -1 to 1. The first one is identified with the character σ_1 of $\mathbb{Q}(x)^*/U_v(\mathbb{Q}(x)^*)^2$ which maps all cosets $fU_v(\mathbb{Q}(x)^*)^2$ for which $v(f)$ is odd to -1, and the second one with the character σ_2 of $\mathbb{Q}(x)^*/U_v(\mathbb{Q}(x)^*)^2$ which maps all cosets $fU_v(\mathbb{Q}(x)^*)^2$ for which $v(f)$ is odd to 1. In turn, σ_1 is identified with a character ρ_1 of $\mathbb{Q}(x)^*$ trivial on $U_v(\mathbb{Q}(x)^*)^2$, which maps p to -1, and σ_2 with a similar character ρ_2 of $\mathbb{Q}(x)^*$, mapping p to 1. Finally, ρ_1 gives rise to an ordering Q_ξ^- of $\mathbb{Q}(x)$, which, in terms of polynomials, can be described as follows: if $f = p^k g$, $p \nmid g$, $f, g \in \mathbb{Q}[x]$, then

$$f \in Q_\xi^- \Leftrightarrow [(g(\xi) > 0 \wedge k \text{ is even}) \vee (g(\xi) < 0 \wedge k \text{ is odd})],$$

and ρ_2 to an ordering $\mathbb{Q}_\xi^+$ such that

$$f \in Q_\xi^+ \Leftrightarrow [g(\xi) > 0].$$

All nontrivial valuations v of $\mathbb{Q}(x)$ trivial on $\mathbb{Q}$ and such that $\mathbb{Q}[x] \subset A_v$ are induced by irreducible polynomials in the way described above ([**Pre84**, Proposition 7.1]). Suppose that v is a valuation with formally real residue field with $\mathbb{Q}[x] \not\subseteq A_v$. Then $\mathbb{Q} \subset A_v$ and, consequently, $x^{-1} \in A_v$; one shows that, for $\frac{f}{g} \in \mathbb{Q}(x)$, $f, g \in \mathbb{Q}[x]$, $v(\frac{f}{g}) = \rho \cdot (-\deg f + \deg g)$, where $\rho \in \mathbb{Z}$ is the positive integer such that $v(x^{-1}) = \rho$. This valuation induces two orderings Q_∞^+ and Q_∞^-, which, in terms of polynomials, can be described as follows: if $f \in \mathbb{Q}[x]$, then

$$f \in Q_\infty^+ \Leftrightarrow [f \text{ has a positive leading coefficient}],$$

$$f \in Q_\infty^- \Leftrightarrow [(\deg f \text{ is even and } f \text{ has a positive leading coefficient}) \vee$$
$$(\deg f \text{ is odd and } f \text{ has a negative leading coefficient})].$$

Finally, if P is an ordering of $\mathbb{Q}(x)$, consider the set

$$B = \{f \in \mathbb{Q}(x) : n + f, n - f \in P \text{ for some integer } n \geq 1\}.$$

B is a valuation ring of $\mathbb{Q}(x)$ [**Mar96**, Theorem 1.3.1]. If $B = \mathbb{Q}(x)$, then P is Archimedean and comes from an embedding $\mathbb{Q}(x) \hookrightarrow \mathbb{R}$, where x is mapped onto some transcendental number. Otherwise B is associated with some nontrivial valuation v, and P is one of the orderings compatible with v and described above.

1.3. Quadratic forms and axioms for spaces of orderings

Let K be a field, char$K \neq 2$, and let T be a proper preordering of K. Define the set:

$$X_T = \{P : P \text{ is an ordering of } K, P \supset T\}.$$

For example, if $v : K \to G \cup \{\infty\}$ is a valuation with formally real residue field, let

$$S = \{1 + a : a \in M_v\},$$

and consider the preordering $\Sigma K^2[S]$. The set $X_{\Sigma K^2[S]}$ is in this case equal to the set X_v (see [**Lam81**, Theorem 2.3]). Clearly $X_{\Sigma K^2}$ is just X_K.

Observe that the set $T^* = T \setminus \{0\}$ is a subgroup of K^*: if $t \in T^*$, then $\frac{1}{t} = (\frac{1}{t})^2 t \in T$. Thus K^*/T^* is a well defined group, which will be denoted by G_T. G_T is naturally identified with a subgroup of the group $\{-1,1\}^{X_T}$ of all functions from X_T to $\{-1,1\}$, with the multiplication defined pointwise: $a \in K^*$ gives a rise to the function $X_T \ni P \mapsto a(P) \in \{-1,1\}$, where

$$a(P) = \begin{cases} 1, & \text{if } a \in P \\ -1, & \text{if } a \in -P. \end{cases}$$

This correspondence is a homomorphism with kernel equal to T^*; indeed, if $a \notin T$, for some $a \in K^*$, then, by Lemma 1.1.1, there is an ordering $P \in X_T$ such that $a \notin P$, so that $a(P) = -1$. When $T = \Sigma K^2$, we shall write G_K instead of G_T, and when T is a preordering underlying all orderings compatible with a valuation v, that is $T = \Sigma K^2[S]$, $S = \{1 + a : a \in M_v\}$, we shall denote G_T by G_v.

A **quadratic form** with entries in G_T is an n-tuple $\phi = (\underline{a_1}, \ldots, \underline{a_n})$, $\underline{a_1}, \ldots, \underline{a_n} \in G_T$, and the number n is called the **dimension** of ϕ. Let $\underline{a_i} = a_i T^*$, $a_i \in K^*$, $i \in \{1, \ldots, n\}$. The integer $\phi(P) = \sum_{i=1}^n a_i(P) \in \mathbb{Z}$ is said to be the **signature** of ϕ at P, where $P \in X_K$. We say that an element $\underline{b} \in G_T$, $\underline{b} = bT^*$, is **represented** by ϕ if, for some $t_1, \ldots, t_n \in T$:

$$b = t_1 a_1 + \ldots + t_n a_n,$$

and we denote by $D(\phi)$ or by $D(\underline{a_1}, \ldots, \underline{a_n})$ the **value set** of all elements represented by ϕ. With a slight abuse of the notation we shall use the same symbol to denote an element of K^*, a coset in G_T, and a function in $\{-1, 1\}^{X_T}$; in particular, in the future we will not underline entries of a quadratic form to stress that we deal with cosets.

We shall say that the form $\phi = (a_1, \ldots, a_n)$ is **isotropic**, if there exist $t_1, \ldots, t_n$ not all equal zero such that

$$0 = t_1 a_1 + \ldots + t_n a_n$$

(we point out the apparent ambiguity in notation mentioned before: entries $a_1, \ldots, a_n$ of ϕ are considered as cosets, whilst terms in the equation $0 = t_1 a_1 + \ldots + t_n a_n$ are just elements of the field). Observe that, for a binary form (a_1, a_2), this just means that $a_1 = -a_2$, for a_1, a_2 viewed as cosets. Thus a binary form which is isotropic is of the shape $(a, -a)$ for $a \in G_T$.

Obviously, for $a \in G_T$, $D(a) = \{a\}$. For $n \geq 3$

$$b \in D(a_1, \ldots, a_n) \Leftrightarrow b \in D(a_1, c) \text{ for some } c \in D(a_2, \ldots, a_n);$$

this is clear when $b = t_1 a_1 + \ldots + t_n a_n$ for some $t_1, \ldots, t_n \in T$, and $t_2 a_2 + \ldots + t_n a_n \neq 0$. If $t_2 a_2 + \ldots + t_n a_n = 0$, we have $b = t_1 a_1 = t_1 a_1 + 0c$ for any c and, since we do not allow $c = 0$, we may take c arbitrary in $D(a_2, \ldots, a_n)$. Thus the study of value sets reduces to the 2-dimensional case, where we shall use the following characterization, which does not refer to the addition in K ([**Mar96**, Lemma 2.1.2]):

1.3.1. LEMMA. $D(a_1, a_2) = \{b \in G_T : \forall P \in X_T(b(P) = a_1(P) \vee b(P) = a_2(P))\}$.

PROOF. Let $b \in K^*$, $b = t_1a_1 + t_2a_2$, $t_1, t_2 \in T$, and let $P \in X_T$. If $a_1(P) = -a_2(P)$, then, clearly, $b(P) = a_1(P)$ or $b(P) = a_2(P)$. Otherwise, assume that $a_1(P) = a_2(P) = 1$. Then the equation $b = t_1a_1 + t_2a_2$ forces $b(P) = 1$. Similarly, when $a_1(P) = a_2(P) = -1$, then $b(P) = -1$.

Conversely, assume that, for each $P \in X_T$, $b(P) = a_1(P)$ or $b(P) = a_2(P)$. We want to show that $b \in Ta_1 + Ta_2$ or, in other words, that $\frac{b}{a_1} \in T + T\frac{a_2}{a_1}$. Suppose, *a contrario,* that $\frac{b}{a_1} \notin T + T\frac{a_2}{a_1}$, and consider the preordering $T' = T + T\frac{a_2}{a_1}$. By Lemma 1.1.1, there is an ordering P such that $T' \subset P$ and $\frac{b}{a_1} \notin P$. Since $T \subset T'$ and $\frac{a_2}{a_1} \in T'$, this implies $P \in X_T$, $\frac{a_2}{a_1}(P) = 1$ and $\frac{b}{a_1}(P) = -1$ which, in turn, forces $a_1(P) = a_2(P)$ and $b(P) = -a_1(P)$. $\square$

The above considerations lead us to the definition of an abstract space of orderings. A **space of orderings** is a pair (X, G), where X is a non-empty set, G is a subgroup of $\{-1, 1\}^X$ containing the constant function -1, and such that the following axioms are satisfied:

(A1): $\forall x, y \in X[(x \neq y) \Rightarrow \exists a \in G(a(x) \neq a(y))]$.

We can view elements of X as characters on G: a natural embedding of X into the character group $\chi(G)$ is obtained by identifying $x \in X$ with the character $a \mapsto a(x)$. If $a, b \in G$, we define the value set $D(a, b)$ as follows:

$$D(a, b) = \{c \in G : \forall x \in X(c(x) = a(x) \vee c(x) = b(x))\}.$$

With those remarks we can state the remaining two axioms:

(A2): If $x \in \chi(G)$ satisfies $x(-1) = -1$, and if

$$\forall a, b \in \ker x \ (D(a, b) \subset \ker x),$$

then x is in the image of the natural embedding $X \hookrightarrow \chi(G)$.

(A3): For $a_1, a_2, a_3 \in G$, if $b \in D(a_1, c)$ for some $c \in D(a_2, a_3)$, then $b \in D(d, a_3)$ for some $d \in D(a_1, a_2)$.

Not surprisingly, if T is a proper preordering in a formally real field K, then the pair (X_T, G_T) is a space of orderings ([**Mar96**, Theorem 2.1.4]). In particular, taking $T = \Sigma K^2$, $(X_K, K^*/(\Sigma K^2)^*)$ is a space of orderings as long as K is formally real.

1.4. Quadratic forms in spaces of orderings

Definitions of forms, dimensions, signatures, and value sets in this general setting are defined exactly as before: for a fixed space of orderings (X, G), a **quadratic form** with entries in G is an n-tuple $\phi = (a_1, \ldots, a_n)$, $a_1, \ldots, a_n \in G$, the number n is called the **dimension** of ϕ, the integer $\phi(x) = \sum_{i=1}^n a_i(x)$ the **signature** of ϕ at x, and, finally, the value set of a form is defined by induction: for a one-dimensional form (a), the value set $D(a)$ is just $\{a\}$, for a binary form the definition has been already stated, and for an n-dimensional form $(a_1, \ldots, a_n)$, $n \geq 3$, it is as follows:

$$D(a_1, \ldots, a_n) = \bigcup_{b \in D(a_2, \ldots, a_n)} D(a_1, b).$$

For two quadratic forms $\phi = (a_1, \ldots, a_n)$ and $\psi = (b_1, \ldots, b_m)$, and for an element $c \in G$, we define the **direct sum**, **scalar product** and **tensor product** as, respectively:

$$\phi \oplus \psi = (a_1, \ldots, a_n, b_1, \ldots, b_m), \quad c\phi = (ca_1, \ldots, ca_n), \quad \phi \otimes \psi = a_1\psi \oplus \ldots \oplus a_n\psi.$$

We shall also denote by $k \times \psi$ the form $\underbrace{\psi \oplus \ldots \oplus \psi}_{k}$. Forms of the shape $(1, a_1) \otimes \ldots \otimes (1, a_n)$ will be called **Pfister forms**, and denoted by $((a_1, \ldots, a_n))$. Some properties of value sets are summarized in the following lemma ([**Mar96**, Theorem 2.2.1 and Corollary 2.2.2]):

1.4.1. Lemma. (1) *$D(\phi)$ does not depend on the order of the entries of ϕ,*

(2) *$D(c\phi) = cD(\phi)$, $c \in G$,*

(3) *$c \in D(\phi \oplus \psi)$ if and only if $c \in D(a, b)$, for some $a \in D(\phi)$ and $b \in D(\psi)$,*

(4) *$D(\psi \oplus \psi) = D(\psi)$.*

The relation of **isometry** of two forms, denoted $\phi \cong \psi$, is defined by analogy to the isometry of quadratic forms over formally real fields with two square classes: two forms of the same dimension ϕ and ψ are isometric if and only if, for every $x \in X$, $\phi(x) = \psi(x)$.

Some properties of the isometry are listed below ([**Mar96**, Theorems 2.2.3 and 2.2.5]):

1.4.2. LEMMA. (1) *$b \in D(\phi)$ if and only if $\phi \cong (b, c_2, \ldots, c_n)$ for some $c_2, \ldots, c_n \in G$, where $n = \dim \phi$,*

(2) *if $\phi \cong \psi$, then $D(\phi) = D(\psi)$,*

(3) *if $b_i = a_{\pi(i)}$, $i \in \{1, \ldots, n\}$, for some permutation π of the set $\{1, \ldots, n\}$, then $(a_1, \ldots, a_n) \cong (b_1, \ldots, b_n)$,*

(4) *if $\phi \cong \psi$, then $c\phi \cong c\psi$, $c \in G$,*

(5) *the relation $\cong$ is transitive,*

(6) *if, for any forms ϕ_1, ϕ_2, ψ_1, ψ_2, $\phi_1 \cong \phi_2$ and $\psi_1 \cong \psi_2$, then both $\phi_1 \oplus \psi_1 \cong \phi_2 \oplus \psi_2$ and $\phi_1 \otimes \psi_1 \cong \phi_2 \otimes \psi_2$,*

(7) *(Witt cancellation theorem) if, for any forms ϕ_1, ϕ_2, ψ_1, ψ_2, $\phi_1 \oplus \psi_1 \cong \phi_2 \oplus \psi_2$ and $\psi_1 \cong \psi_2$, then also $\phi_1 \cong \phi_2$,*

(8) *(alternate description of isometry) if $\phi = (a_1, \ldots, a_n)$ and $\psi = (b_1, \ldots, b_n)$, then*

$$\phi \cong \psi \Leftrightarrow \exists a, b, c_3, \ldots, c_n \in G[(a_2, \ldots, a_n) \cong (a, c_3, \ldots, c_n) \wedge$$
$$\wedge \quad (a_1, a) \cong (b_1, b) \wedge (b_2, \ldots, b_n) \cong (b, c_3, \ldots, c_n)].$$

Finally, we introduce the notion of the **isotropy**. A form ϕ will be called isotropic, if there exists a form ψ such that $\phi \cong (-1, 1) \oplus \psi$. Otherwise, ϕ will be called **anisotropic**.

1.5. Subspaces of spaces of orderings

As before, let (X, G) be a space of orderings. In X we introduce a natural topology, called the **Harrison topology,** as the weakest topology such that the functions $a : X \to \{-1, 1\}$, $a \in G$, are continuous, given that $\{-1, 1\}$ has the discrete topology. In other words, the sets

$$U(a) = \{x \in X : a(x) = 1\}, \quad a \in G,$$

are clopen and form a subbasis for the topology on X, and the sets

$$U(a_1, \dots, a_n) = \bigcap_{i=1}^{n} U(a_i)$$

form a basis for the topology on X. X endowed with the Harrison topology is a Boolean space (that is: compact, Hausdorff, and totally disconnected) ([**Mar96**, Theorem 2.1.5]).

A subset $Y \subset X$ will be called a **subspace** of (X, G), if Y is expressible in the form $\bigcap_{a \in S} U(a)$ for some, not necessarily finite, subset $S \subset G$. For any subspace Y we will denote by $G|_Y$ the group of all restrictions $a|_Y$, $a \in G$. Not surprisingly, the pair $(Y, G|_Y)$ is a space of orderings itself ([**Mar96**, Theorem 2.4.3]).

Let K be a formally real field and consider the space of orderings (X_K, G_K). Subspaces of (X_K, G_K) are of the form X_T, where $T \subsetneq K$ is a proper preordering in K. Indeed, if $Y \subset X_K$ is a subspace, $Y = \bigcap_{a \in S} U(a)$, then $Y = X_T$, where $T = \Sigma K^2[S]$. Conversely, if T is a proper preordering, then $X_T = \bigcap_{a \in T^*} U(a)$. Clearly $G_K|_{X_T} \cong G_T$.

Consider a form $\phi = (b_1, \dots, b_k)$ in an arbitrary space of orderings (X, G). When we refer to the form ϕ in a subspace Y, we mean the form

$$\phi|_Y = (b_1|_Y, \dots, b_k|_Y).$$

However, to avoid the use of lengthy and illegible notation, we shall simply write ϕ for both the form in (X, G) and the form in $(Y, G|_Y)$, as long as it is clear with which space we work. Similarly, when we refer to isometry of two same-dimensional forms ϕ and ψ in Y, we mean isometry in this particular space, that is:

$$\forall x \in Y (\phi(x) = \psi(x)),$$

and when we refer to the value set $D(\phi)$ of the form ϕ in Y, we mean the set $D(\phi|_Y)$ in the space $(Y, G|_Y)$. This set will sometimes be denoted by $D_Y(\phi|_Y)$ – however, for the sake of simplicity, we will avoid using separate notation for most of the time.

1.6. Fans

Let G be a multiplicative group of exponent 2 with an element $e \neq 1$ (to play the role of the constant function -1). The pair (X, G), where

$$X = \{x \in \chi(G) : x(e) = -1\}.$$

will be called a **fan**. Elements of G can be viewed as functions on X by defining $a(x) = x(a)$ for $a \in G$, $x \in X$. Any fan is a space of orderings ([**Mar96**, Theorem 3.1.1]).

Let K be a formally real field, and let v be a valuation on K whose residue field K_v is uniquely ordered, or has precisely two orderings. Then the subspace (X_v, G_v) of the space (X_K, G_K) is an example of a fan; in general, if the space of orderings of the residue field is a fan, then (X_v, G_v) is also a fan ([**Mar96**, Theorem 3.6.1]).

Fans can be characterized in many ways. Later in our work we will need the following lemma ([**Mar96**, Theorem 3.1.2]):

1.6.1. Lemma. *Let (X, G) be a space of orderings. Then the following are equivalent:*

(1) *(X, G) is a fan,*
(2) *if $a \neq -1$, then $D(1, a) = \{1, a\}$,*
(3) *if $a_i a_j \neq -1$, for $i \neq j$, $a_1, \ldots, a_n \in G$, then $D(a_1, \ldots, a_n) = \{a_1, \ldots, a_n\}$.*

We will be mostly dealing with finite fans, and thus we need to know how to recognize when a finite space of orderings is a fan. Let (X, G) be a finite space of orderings. Since G is of exponent 2, we can view G as a vector space over the field $\mathbb{F}_2$, and $\chi(G)$ as a dual space to G. Thus G is a direct sum of cyclic groups of order 2, and so is $\chi(G)$; since G is finite, $G \cong \chi(G)$. If we view elements of X as characters, we have $\bigcap_{x \in X} \ker x = \{1\}$, and thus we can find some smallest subset $\{x_1, \ldots, x_n\}$ of X such that $\bigcap_{i=1}^n \ker x_i = \{1\}$. Any such set will be called a **minimal generating set** of (X, G). Main properties of fans and minimal generating sets are summarized in the following lemma ([**Mar96**, Theorem 3.1.3]):

1.6.2. Lemma. *Let (X, G) be a space of orderings with a minimal generating set $\{x_1, \ldots, x_n\}$.*

(1) $|G| = 2^n$,

(2) $\{x_1, \ldots, x_n\}$ *is an* $\mathbb{F}_2$ *basis for the character group* $\chi(G)$*; in particular:*

$$\forall x \in X \exists e_1, \ldots, e_n \in \{0,1\}(x = \prod_{i=1}^{n} x_i^{e_i}),$$

(3) *if* $x \in X$ *and* $x = \prod_{i=1}^{n} x_i^{e_i}$*, for some* $e_1, \ldots, e_n \in \{0,1\}$*, then:*

$$\sum_{i=1}^{n} e_i \equiv 1 \mod 2;$$

in particular, $n \leq |X| \leq 2^{n-1}$,

(4) (X, G) *is a fan if and only if* X *consists of all products* $\prod_{i=1}^{n} x_i^{e_i}$ *such that* $\sum_{i=1}^{n} e_i \equiv 1 \mod 2$*; in particular,* $|X| = 2^{n-1}$.

It follows that if (X, G) has a finite generating set, then (X, G) is finite. If $\{x_1, \ldots, x_n\}$ is a minimal generating set, then the set of elements $\{a_1, \ldots, a_n\}$ of G such that

$$a_i(x_j) = x_j(a_i) = \begin{cases} 1, & \text{if } i \neq j, \\ -1, & \text{if } i = j \end{cases}$$

will be called the **dual basis**. By evaluating at each x_j we check that $a_1 a_2 \ldots a_n = -1$.

We shall investigate a few simple cases of minimal generating sets. For a minimal generating set consisting of one or two elements, the generated space has only one or two elements, respectively. In both cases such a space is a fan – we shall call it a **trivial** fan. A one element space shall be also called a **singleton** space.

A space X generated by three elements x_1, x_2, x_3 can consist of three or four elements. If $|X| = 4$, then X is a fan. By Lemma 1.6.2, it contains the character $x_1 x_2 x_3$, and consists of 4 distinct elements, $y_1, \ldots, y_4$, such that

$$\forall a \in G(\prod_{i=1}^{4} a(y_i) = 1).$$

The 4 element fans are especially important.

1.7. The stability index

As before, let (X, G) be a space of orderings. The **stability index** of (X, G), denoted $\mathrm{stab}(X, G)$, is the maximal integer n such that there exists a fan (Y, H) with $Y \subset X$ and $|Y| = 2^n$, or ∞ if no such finite n exists. We shall frequently use the following equivalent definition of the stability index ([**Mar96**, Theorem 3.4.2]):

1.7.1. LEMMA. *For a space of orderings (X, G) and $k \geq 1$, the following two conditions are equivalent:*

(1) $\mathrm{stab}(X, G) \leq k$*,*

(2) *every basic set $V \subset X$ (in the Harrison topology) is expressible as $V = U(a_1, \dots, a_k)$ for some $a_1, \dots, a_k \in G$.*

Spaces of stability index zero are just the singleton spaces. Spaces of stability index equal at most one are said to satisfy the **strong approximation property**. The name is explained by the following lemma ([**Mar96**, Theorem 3.3.1]):

1.7.2. LEMMA. *For a space of orderings (X, G) the following three conditions are equivalent:*

(1) $\mathrm{stab}(X, G) \leq 1$*,*

(2) $G = \mathcal{C}(X, \{-1, 1\})$*, where $\mathcal{C}(X, \{-1, 1\})$ denotes the set of all continuous functions $f : X \to \{-1, 1\}$,*

(3) *for each pair of disjoint closed sets Y_1, Y_2 in X, there is an element $a \in G$ such that $a > 0$ on Y_1, whilst $a < 0$ on Y_2,*

(4) *every closed subset of X is a subspace of X.*

Computing the stability index of a space of orderings is usually a complicated matter. For what we need in the further course of our work, we shall quote the following result ([**ABR**, Proposition VI.3.2 and Proposition VI.3.5]):

1.7.3. LEMMA. (1) *If F is a formally real algebraic function field over a real closed field K, and $d = \mathrm{trdeg}(F : K)$ is its transcendence degree, then $\mathrm{stab}(X_F, G_F) = d$.*

(2) *If F is a formally real algebraic function field over $\mathbb{Q}$, and $d = \mathrm{trdeg}(F : \mathbb{Q})$, then* $\mathrm{stab}(X_F, G_F) = d + 1$.

In particular, $\mathrm{stab}(X_{\mathbb{R}(x)}, G_{\mathbb{R}(x)}) = 1$, $\mathrm{stab}(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)}) = 2$, $\mathrm{stab}(X_F, G_F) = 2$, for F a formally real function field of a conic section over $\mathbb{Q}$, and $\mathrm{stab}(X_{\mathbb{R}(x,y)}, G_{\mathbb{R}(x,y)}) = 2$.

1.8. Group extensions and direct sums

Let $(\overline{X}, \overline{G})$ be a space of orderings. We say that a pair (X, G) is a **group extension** of $(\overline{X}, \overline{G})$, if G is an extension of the group $\overline{G}$ (that is, $\overline{G}$ is a subgroup of G), and X is the subset of the set $\chi(G)$ consisting of all characters x on G such that $x|_{\overline{G}} \in \overline{X}$. Any group extension of a space of orderings is a space of orderings ([**Mar96**, Theorem 4.1.1]).

We shall be interested in describing value sets of quadratic forms in group extensions. Since G has exponent 2 and can be viewed as a vector space over $\mathbb{F}_2$, G can be decomposed as a direct product $G = \overline{G} \times H$ for some group H (note that this decomposition is never unique except for the trivial case when $\overline{G} = G$), and, consequently, $X = \overline{X} \times \chi(H)$, where $\chi(H)$ denotes the group of characters of H. We will view H as a vector space over $\mathbb{F}_2$. If ϕ is a form with entries in G, then ϕ can be represented as a direct sum $\phi = h_1\phi_1 \oplus \ldots \oplus h_s\phi_s$, for some distinct $h_1, \ldots, h_s \in H$, and for forms $\phi_1, \ldots, \phi_s$ with entries in $\overline{G}$. The forms $\phi_1, \ldots, \phi_s$ will be called the **residue forms** of ϕ. By [**Mar96**, Theorem 4.1.1], the value set of ϕ is:

$$D(\phi) = \begin{cases} \bigcup_{i=1}^{s} h_i D(\phi_i), & \text{if } \phi_1, \ldots, \phi_s \text{ are anisotropic,} \\ G, & \text{if some of } \phi_1, \ldots, \phi_s \text{ are isotropic.} \end{cases}$$

As a special case we shall consider a form $(1, a)$, where $a \in G$. According to the above, the value set $D(1, a)$ is the following one:

$$D(1,a) = \begin{cases} \{1, a\}, & \text{if } a \notin \overline{G}, \\ G, & \text{if } a = -1, \\ D_{\overline{X}}(1, a), & \text{if } a \in \overline{G} \text{ and } a \neq -1, \end{cases}$$

where $D_{\overline{X}}(1, a)$ denotes the value set considered in the space $(\overline{X}, \overline{G})$.

Suppose that (X, G) is a space of orderings, and consider the group:

$$\tilde{G} = \{a \in G : \forall y \in \chi(G)[yX = X \Rightarrow y(a) = 1]\}$$

(here we identify X with its image under the natural embedding $X \hookrightarrow \chi(G)$), and let $\tilde{X}$ denote the image of X under the restriction mapping $\chi(G) \supset X \ni x \mapsto x|_{\tilde{G}} \in \chi(\tilde{G})$. Then the pair $(\tilde{X}, \tilde{G})$ is a space of orderings and (X, G) is a group extension of $(\tilde{X}, \tilde{G})$ ([**Mar96**, Theorem 4.1.3]). Moreover, $(\tilde{X}, \tilde{G})$ is minimal in the sense that if (X, G) is a group extension of some space $(\overline{X}, \overline{G})$, then $\tilde{G} \subset \overline{G}$. $(\tilde{X}, \tilde{G})$ will be called the **residue space** of (X, G).

We shall now introduce the notion of direct sums of spaces of orderings. A pair (X, G) will be called the **direct sum** of the spaces of orderings $(X_1, G_1), \ldots, (X_n, G_n)$, denoted $(X, G) = (X_1, G_1) \oplus \ldots \oplus (X_n, G_n)$, when X is the disjoint union of $X_1, \ldots, X_n$, and G consists of all functions $a : X \to \{-1, 1\}$ such that $a|_{X_i} \in G_i$, $i \in \{1, \ldots, n\}$. Any direct sum of spaces of orderings is a space of orderings ([**Mar96**, Theorem 4.1.1]). We will also need the following lemma ([**Mar96**, Theorem 4.1.2]):

1.8.1. Lemma. *Let (X, G) be a space of orderings, and let $X_1, \ldots, X_n$ be a partition of X into closed sets.*

(1) *If each X_i is a subspace of (X, G) and $(X, G) = (X_1, G_1) \oplus \ldots \oplus (X_n, G_n)$, then each nontrivial fan in X lies in some X_i.*

(2) *If each 4-element fan in X lies in some X_i, then each X_i is a subspace of (X, G), and $(X, G) = (X_1, G_1) \oplus \ldots \oplus (X_n, G_n)$.*

For a space of orderings (X, G) we define the **connectivity relation** $\sim$ on X: for $x_1, x_2 \in X$, $x_1 \sim x_2$ if and only if either $x_1 = x_2$, or there exist $x_3, x_4 \in X$ such that $\{x_1, x_2, x_3, x_4\}$ is a 4-element fan in X. $\sim$ is an equivalence relation on X ([**Mar96**, Theorem 4.6.1]), and the equivalence classes of $\sim$ will be called the **connected components** of X.

1.9. Chain length and the Structure and Isotropy Theorems

We shall conclude this brief summary of the theory of spaces of orderings with the notion of the chain length. The **chain length** of a space of orderings, denoted $\mathrm{cl}(X, G)$, is the maximal number d such that there exist $a_1, \ldots, a_d \in G$ with $U(a_0) \subsetneq \ldots \subsetneq U(a_d)$, or ∞ if there is no integer with that property. Note that $U(a) \subset U(b)$ is equivalent to $b \in D(1, a)$, $a, b \in G$. The fundamental result concerning spaces of orderings of finite chain length is the following one ([**Mar96**, Theorem 4.2.2]):

1.9.1. THEOREM (Structure Theorem). *Every space of orderings of finite chain length is built up, recursively, in an essentially unique way, from singleton spaces, using the direct sum and group extension constructions.*

Clearly every finite space of orderings has finite chain length. Next, we note:

1.9.2. THEOREM (Isotropy Theorem). *Let (X, G) be a space of orderings and let ϕ be an anisotropic form with entries in G. There exists a finite subspace Y of X such that ϕ is anisotropic in $(Y, G|_Y)$.*

This result, originally proven in [**Mar80-2**], has a strong, generalized version ([**Mar84**]):

1.9.3. THEOREM (Extended Isotropy Theorem). *Let (X, G) be a space of orderings and let $\phi_1, \ldots, \phi_n$ be quadratic forms with entries in G. If $\bigcap_{i=1}^n D(\phi) = \emptyset$, then there exists a finite subspace Y of X such that $\bigcap_{i=1}^n D(\phi) = \emptyset$ in $(Y, G|_Y)$.*

1.10. The language L_{SG} and special groups

In the last section of this chapter we shall recall the notion of the first order language L_{SG} and a theory in this language, which will serve as an abstract framework for studying the theory of spaces of orderings. The **language** L_{SG} consists of a quaternary relation symbol $\cong$ called isometry, a functional symbol $\cdot$ called multiplication, and two constants -1 and 1. We use the usual set of logical symbols: $\neg, \rightarrow$, a set of individual variables V, the quantifier $\forall$ and the identity symbol $=$. We define terms T by induction as the smallest set containing individual variables and constants, which is closed under the functional symbol. For terms $t_1, \ldots, t_4 \in T$ we define atomic formulae to be of the form either $t_1 = t_2$ or $(t_1, t_2) \cong (t_3, t_4)$.

In this language we build the **theory of reduced special groups** as the set of sentences:

(1) $\cdot$ is a group multiplication,

(2) $\forall a(a \cdot a = 1)$,

(3) $\cong$ is an equivalence relation,

(4) $\forall a, b[(a, b) \cong (b, a)]$,

(5) $\forall a[(a, -a) \cong (-1, 1)]$,

(6) $\forall a, b, c, d[(a, b) \cong (c, d) \rightarrow a \cdot b = c \cdot d]$,

(7) $\forall a, b, c, d[(a, b) \cong (c, d) \rightarrow (a, -c) \cong (-b, d)]$,

(8) $\forall a, b, c, d\{[(a, b) \cong (c, d)] \rightarrow \forall x[(x \cdot a, x \cdot b) \cong (x \cdot c, x \cdot d)]\}$,

(9) $\forall a[(a, a) \cong (1, 1) \leftrightarrow a = 1]$,

(10) $\cong_3$ is transitive, where

$$(a_1, a_2, a_3) \cong_3 (b_1, b_2, b_3) \Leftrightarrow \exists a, b, c_3[(a_1, a) \cong (b_1, b) \wedge (a_2, a_3) \cong (a, c_3) \wedge (b_2, b_3) \cong (b, c_3)].$$

Since we are more used to the value set notation than to the isometry relation, we shall introduce the following abbreviation:

$$a \in D(b, c) \Leftrightarrow (b, c) \cong (a, abc).$$

In view of the above axioms, we see that

$$[(a, b) \cong (c, d)] \Leftrightarrow [ab = cd \wedge ac \in D(1, cd)]$$

and thus we may interchange the quaternary relation $(a, b) \cong (c, d)$ with the ternary one $a \in D(b, c)$. Furthermore, since $a \in D(b, c)$ if and only if $ab \in D(1, bc)$, and since $a = b$ if and only if $ab \in D(1, 1)$, we shall generally accept $a \in D(1, b)$ as atomic formulae.

Clearly any model of the theory of reduced special groups shall be called a **reduced special group**. Since the language of special groups differs from the language of groups, we shall denote special groups by $(G, \cong, -1)$. An **SG-morphism** is a group homomorphism f between two reduced special groups $(G, \cong, -1)$ and $(H, \cong, -1)$ such that $f(-1) = -1$ and

$$\forall a, b, c, d \in G\{[(a, b) \cong (c, d)] \Rightarrow [(f(a), f(b)) \cong (f(c), f(d))]\},$$

or, equivalently,

$$\forall a, b \in G[(a \in D(1, b)) \Rightarrow (f(a) \in D(1, f(b)))].$$

For a reduced special group $(G, \cong, -1)$ denote by X_G the set of all SG-morphisms of G into the two-element reduced special group $\mathbb{Z}_2 = \{-1, 1\}$. One shows that (X_G, G) is a space of orderings ([**DicMir00**, Propositon 3.10]). Moreover, for a space of orderings (X, G), $(G, \cong, -1)$ is a reduced special group (with the usual meaning of $\cong$ and -1) ([**DicMir00**, Proposition 3.11]), and the two correspondences:

$$(G, \cong, -1) \mapsto (X_G, G) \text{ and } (X, G) \mapsto (G, \cong, -1)$$

are reciprocal to each other ([**DicMir00**, Proposition 3.14]).

CHAPTER 2

General properties of pp formulae

In this chapter we formally introduce the notion of pp formulae and state the pp conjecture. Most of the material presented in Section 1 is drawn from [**Mar02**]. In Section 2 we prove that the pp conjecture is preserved in subspaces, and use this result to describe the behavior of the pp conjecture in direct sums and group extensions. Lemma 2.2.2 appeared in print for the first time in [**AstTre05**], and Theorem 2.2.1 is proven in [**AstTre05**] using other methods – the proof given in our work has not been published before; both our proof and the proof by Astier and Tressl use Lemma 2.2.2. Product free and one-related pp formulae, which we discuss in Section 3, were introduced in [**Mar06**], where Theorem 2.3.1 was also proven – however, the proof of this theorem is a modification of the proof of the Extended Isotropy Theorem and, in turn, the Isotropy Theorem, which trace back to the works [**Mar80-2**] and [**Mar84**]. Theorem 2.2.9 was proven in [**DicMarMir05**]. Finally, in Section 4 we investigate products of value sets of quadratic forms, following Section 3 of [**Mar02**]. Proofs given in this section are not new, yet we decided to include them in our work for the sake of completeness.

2.1. Basic definitions

In this section we shall formally state the definition of a positive primitive formula, and prove some elementary properties of pp formulae. Let L be a first-order language (we can think of L as the language of special groups L_{SG}). A first-order formula in the language L with parameters $\underline{a} = (a_1, \dots, a_k)$ is said to be **positive primitive** (pp for short), if it is of the form $\exists \underline{t} \Psi(\underline{t}, \underline{a})$, where $\underline{t} = (t_1, \dots, t_n)$, and $\Psi(\underline{t}, \underline{a})$ is a finite conjunction of atomic formulae.

We are interested in pp formulae arising in the theory of spaces of orderings. Let (X, G) be a space of orderings. A pp formula $P(\underline{a})$ with n quantifiers and k parameters in G is expressible as

$$\exists \underline{t} \bigwedge_{j=1}^{m} p_j(\underline{t}, \underline{a}) \in D(1, q_j(\underline{t}, \underline{a})),$$

where $\underline{t} = (t_1, \ldots, t_n)$, $\underline{a} = (a_1, \ldots, a_k)$, for $a_l \in G$, $l \in \{1, \ldots, k\}$, and $p_j(\underline{t}, \underline{a})$, $q_j(\underline{t}, \underline{a})$ are $\pm$ products of some of the t_i's and a_l's, $i \in \{1, \ldots, n\}$, $l \in \{1, \ldots, k\}$. We shall give a few examples of pp formulae.

The formula "two forms are isometric" is a pp formula. Recall that, for two forms ϕ and ψ of the same dimension d, $\phi \cong \psi$ means that

$$\forall x \in X(\phi(x) = \psi(x)),$$

which, if $\phi = (a_1, \ldots, a_d)$, $\psi = (a'_1, \ldots, a'_d)$, can be equivalently stated (see Lemma 1.4.2) as

$$\begin{aligned} &\exists t, t', t_3, \ldots, t_d \in G[(a_2, \ldots, a_d) \cong (t, t_3, \ldots, t_d) \wedge \\ \wedge \quad &(a_1, t) \cong (a'_1, t') \wedge (a'_2, \ldots, a'_d) \cong (t', t_3, \ldots, t_d)]. \end{aligned}$$

Now, by induction on d, we can easily see that the isometry $\phi \cong \psi$ is expressible as a finite conjunction of atomic formulae preceded by existential quantifiers.

The formula "an element is represented by a form" is a pp formula. Recall that, for a form $\phi = (a_1, \ldots, a_d)$, $d \geq 3$, we have:

$$a \in D(\phi) \Leftrightarrow \exists t \in G[a \in D(a_1, t) \wedge t \in D(a_2, \ldots, a_d)].$$

It is clear that the property of representability by a form is expressible as a pp formula.

Suppose that $P(\underline{a})$ is a pp formula "two forms are isometric" or a pp formula "an element is represented by a form". In both cases the following "local-global principle" is true: if $P(\underline{a})$ holds true in every finite subspace of (X, G), then $P(\underline{a})$ holds true in the whole space (X, G) (while speaking of the formula $P(\underline{a})$ in a subspace Y, we mean the formula obtained from $P(\underline{a})$ by replacing each atom $p \in D(1, q)$ by $p|_Y \in D_Y(1, q|_Y)$). In the case of the formula "two forms are isometric" this is a trivial observation: for two forms ϕ and ψ, $\dim \phi = \dim \psi$, the equation $\phi(x) = \psi(x)$ holds for every $x \in X$ if and only if it holds for x ranging over all singleton subspaces of X. For the formula "an element is represented by a form" the above mentioned "local-global principle" is a consequence of the Isotropy Theorem (Theorem 1.9.2).

In view of the above observations, it is natural to ask the following question, now known as the **pp conjecture**:

For a space of orderings (X, G)*, is it true that a pp formula* $P(\underline{a})$ *with parameters* $\underline{a}$ *in* G*, which holds in every finite subspace of* (X, G)*, necessarily holds in* (X, G)*?*

In addition to the above examples, the pp formula "$\exists t \bigwedge_{i=1}^{\mu} t \in D(\phi_i)$", where $\phi_1, \ldots, \phi_\mu$ are some fixed forms, the pp conjecture also holds true; this fact follows from the Extended Isotropy Theorem (Theorem 1.9.3). It had always seemed unlikely that the conjecture has a positive solution in general, and the aim of our research summarized in this thesis was to construct appropriate counterexamples. In the course of this work we shall first investigate classes of spaces of orderings and types of pp formulae, for which the conjecture holds true, and then discuss two main examples where it fails.

The following lemma, which allows us to restrict our considerations to subspaces minimal subject to the condition that the pp conjecture fails, will be of frequent use. This result is originally due to Astier, and appeared in print in [**Mar02**, Proposition 2.2].

2.1.1. Lemma. *Let* (X, G) *be a space of orderings, let* $P(\underline{a})$ *be a pp formula defined as above. If* $P(\underline{a})$ *fails to hold in* (X, G)*, then there is a subspace* Y *minimal subject to the condition that* $P(\underline{a})$ *fails in* Y*.*

2.2. Behavior of pp formulae in subspaces, direct sums, and group extensions

We shall now proceed to investigate some of the properties of pp formulae.

2.2.1. Theorem. *Let* (X, G) *be a space of orderings, let* Y *be a subspace. If, for every pp formula, the pp conjecture holds in* (X, G)*, then it also holds in* $(Y, G|_Y)$ *for every pp formula.*

The proof given here is based on the following lemma proven by Marshall, which appeared in print for the first time in [**AstTre05**]:

2.2.2. Lemma. *(*[**AstTre05**, Lemma 4]*) Let* $B(n, 0) = 1$ *for* $n \in \mathbb{N}$*, and let*

$$B(n, k) = 2^k 2^{2^{nk} B(n, k-1)}, \text{ if } k \geq 1, n \in \mathbb{N}.$$

Then, for every space of orderings (X, G)*,and for every pp formula* $P(\underline{a})$ *with* n *quantifiers* *and k parameters, if $P(\underline{a})$ fails to hold in $(Z, G|_Z)$, where Z is a finite subspace of (X, G) (or, more generally, is a subspace Z such that $(Z, G|_Z)$ has a finite chain length), then there is a subspace Y of (X, G) such that $P(\underline{a})$ fails to hold in $(Y, G|_Y)$ and $|Y| \leq B(n, k)$.*

The proof of Theorem 2.2.1 is given in [**AstTre05**]; it uses rather general ideas from model theory, and, implicitly, refers to Lemma 2.2.2 as well. We give another proof, which makes use of some basic notions from topology. The reader might wish to consult [**Eng89**] for the definitions of nets, cluster points, and for some of their basic properties. Firstly, we will need two technical lemmas (see [**Mar96**, Theorems 2.4.1, 2.4.4]):

2.2.3. LEMMA. *Let (X, G) be a space of orderings, let $a_1, \ldots, a_n, d \in G$, $n \geq 1$. Then*

$$d \in D[((a_1, \ldots, a_n))] \Leftrightarrow \forall x \in X[a_1(x) = 1 \wedge \ldots \wedge a_n(x) = 1 \Rightarrow d(x) = 1].$$

PROOF. ($\Rightarrow$). We proceed by induction on n. If $n = 1$ then there is nothing to prove. Let $n \geq 2$. By the inductive hypothesis and Lemma 1.4.1 (3):

$$\begin{aligned}
&d \in D[((a_1, \ldots, a_n))] \Leftrightarrow d \in D[((a_1, \ldots, a_{n-1})) \oplus a_n((a_1, \ldots, a_{n-1}))] \\
&\quad \Leftrightarrow \quad \exists t_1, t_2 \in G\{d \in D(t_1, t_2) \wedge t_1 \in D[((a_1, \ldots, a_{n-1}))] \wedge t_2 a_n \in D[((a_1, \ldots, a_{n-1}))]\} \\
&\quad \Rightarrow \quad \exists t_1, t_2 \in G \forall x \in X[(d(x) = t_1(x) \vee d(x) = t_2(x)) \\
&\quad \wedge \quad (a_1(x) = \ldots = a_{n-1}(x) = 1 \Rightarrow t_1(x) = t_2 a_n(x) = 1)] \\
&\quad \Rightarrow \quad \forall x \in X[a_1(x) = \ldots = a_n(x) = 1 \Rightarrow d(x) = 1].
\end{aligned}$$

($\Leftarrow$). Suppose that, for every $x \in X$, if $a_1(x) = \ldots = a_n(x) = 1$, then $d(x) = 1$. Comparing signatures we see that

$$((a_1, \ldots, a_n)) \cong d((a_1, \ldots, a_n))$$

(note that the signature of each side at x is either 2^n or 0). Since $1 \in D[((a_1, \ldots, a_n))]$, we have that $d \in D[d((a_1, \ldots, a_n))]$ and, consequently, $d \in D[((a_1, \ldots, a_n))]$. $\square$

2.2.4. LEMMA. *Let (X, G) be a space of orderings, let $\phi = (b_1, \ldots, b_k)$ be a form in (X, G). If $Y = U(a_1, \ldots, a_n)$, then $c|_Y \in D_Y(\phi_Y)$ if and only if $c \in D[\phi \otimes ((a_1, \ldots, a_n))]$*

Proof. ($\Rightarrow$). If $c|_Y \in D|_Y(\phi|_Y)$ then, by Lemma 1.4.2 (1), $(c|_Y, c_2|_Y, \ldots, c_k|_Y) \cong \phi|_Y$ for some $c_2, \ldots, c_k \in G$, and, comparing signatures, $(c, c_2, \ldots, c_k) \otimes ((a_1, \ldots, a_n)) \cong \phi \otimes ((a_1, \ldots, a_n))$ on X (the signature of each side at x is $\phi|_Y(x) \cdot 2^n$ if $x \in Y$ and 0 otherwise). Since $1 \in D[((a_1, \ldots, a_n))]$, this proves $c \in D[\phi \otimes ((a_1, \ldots, a_n))]$.

($\Leftarrow$). Using $\phi \otimes ((a_1, \ldots, a_n)) = b_1((a_1, \ldots, a_n)) \oplus \ldots \oplus b_k((a_1, \ldots, a_n))$, by Lemma 1.4.1 (3), $c \in D(b_1 d_1, \ldots, b_k d_k)$ for some $d_1, \ldots, d_k \in D[((a_1, \ldots, a_n))]$. Since, by Lemma 2.2.3, $d_i = 1$ on Y, this implies $c|_Y \in D|_Y(\phi|_Y)$. $\square$

We note that the previous two lemmas are, in fact, needed to prove that a subspace of a space of orderings is a space of orderings. We proceed to the proof of the main theorem.

Proof. Let $P(\underline{a}) = \exists \underline{t} \bigwedge_{j=1}^{m} p_j(\underline{a}, \underline{t}) \in D(1, q_j(\underline{a}, \underline{t}))$ be a pp formula, where $\underline{t} = (t_1, \ldots, t_n)$, $\underline{a} = (a_1, \ldots, a_k)$, for $a_l \in G$, $l \in \{1, \ldots, k\}$, and, for $j \in \{1, \ldots, m\}$, $p_j(\underline{t}, \underline{a})$, $q_j(\underline{t}, \underline{a})$ are $\pm$ products of some of the t_i's and a_l's, $i \in \{1, \ldots, n\}$, $l \in \{1, \ldots, k\}$. Suppose that $P(\underline{a})$ holds in every finite subspace of $(Y, G|_Y)$. We shall show that it also holds in $(Y, G|_Y)$, which will complete the proof. There are two cases to consider:

Case 1: Let $Y = U(b_1, \ldots, b_l)$. If Z is a finite subspace of (X, G) then $Z \cap Y$ is a finite subspace of $(Y, G|_Y)$ (possibly empty), so $P(\underline{a})$ holds true in $(Z \cap Y, G|_{Z \cap Y})$. Let

$$P(\underline{a}, Y) = \exists \underline{t} \bigwedge_{j=1}^{m} p_j(\underline{a}, \underline{t}) \in D((1, q_j(\underline{a}, \underline{t})) \otimes ((b_1, \ldots, b_l))).$$

By Lemma 2.2.4, $P(\underline{a}, Y)$ is a pp formula which holds in $(Z, G|_Z)$ if and only if $P(\underline{a})$ holds in $(Z \cap Y, G|_{Z \cap Y})$. Since, for $P(\underline{a}, Y)$, the pp conjecture holds true in (X, G), $P(\underline{a}, Y)$ holds in (X, G), so that $P(\underline{a})$ holds in $(Y, G|_Y)$.

Case 2: Let $Y = \bigcap_{b \in S} U(b)$, for an infinite set $S \subset G$. It suffices to show that, for some finite subset $T \subset S$, $P(\underline{a})$ holds in $\bigcap_{b \in T} U(b)$. Suppose, *a contrario*, that, for every finite subset $T \subset S$, $P(\underline{a})$ fails in $\bigcap_{b \in T} U(b)$. By **case 1**, it follows that, for every finite subset $T \subset S$, there exists a finite subspace Z_T of $\bigcap_{b \in T} U(b)$ such that $P(\underline{a})$ fails in Z_T. By Lemma 2.2.2, there is an integer B with the property that, for every finite subset $T \subset S$, there exists a finite subspace Y_T of $\bigcap_{b \in T} U(b)$ of cardinality at most B, such that $P(\underline{a})$ fails in Y_T. Let $Y_T = \{x_1^T, x_2^T, \ldots, x_B^T\}$. For $i \in \{1, \ldots, B\}$, $\{x_i^T : T \in 2^S, T \text{ is finite}\}$ is a net with entries directed according to the rule

$$x_i^T \geq x_i^{T'} \text{ if and only if } T \supseteq T'.$$

Since X is compact, $\{x_1^T : T \in 2^S, T \text{ is finite}\}$ has a cluster point x_1. Let $\{x_1^{T_1} : T_1 \in \Sigma_1\}$ be a net finer than $\{x_1^T : T \in 2^S, T \text{ is finite}\}$ which converges to x_1, where $\Sigma_1 \subset \{T \in 2^S : T \text{ is finite}\}$. Next, $\{x_2^{T_1} : T_1 \in \Sigma_1\}$ has a cluster point x_2, so let $\{x_2^{T_{12}} : T_{12} \in \Sigma_{12}\}$ be a net finer than $\{x_2^{T_1} : T_1 \in \Sigma_1\}$ which converges to x_2, where $\Sigma_{12} \subset \Sigma_1$. By induction, we will eventually construct the net $\{x_B^{T_{12\ldots B}} : T_{12\ldots B} \in \Sigma_{12\ldots B}\}$ finer than $\{x_B^{T_{12\ldots B-1}} : T_{12\ldots B-1} \in \Sigma_{12\ldots B-1}\}$ which converges to a cluster point x_B of the net $\{x_B^{T_{12\ldots B-1}} : T_{12\ldots B-1} \in \Sigma_{12\ldots B-1}\}$, where $\Sigma_{12\ldots B} \subset \Sigma_{12\ldots B-1}$. Clearly, for every $i \in \{1, \ldots, B\}$, the net

$$\{x_i^{T_{12\ldots B}} : T_{12\ldots B} \in \Sigma_{12\ldots B}\}$$

is finer than $\{x_i^T : T \in 2^S, T \text{ is finite}\}$ and converges to x_i, $i \in \{1, \ldots, B\}$. Let Z be the subspace of (X, G) generated by $x_1, \ldots, x_B$.

We shall show that Z is a subspace of Y; indeed, it suffices to show that all the generators $x_1, \ldots, x_B$ are elements of Y. Fix an arbitrary $i \in \{1, \ldots, B\}$ and $b_0 \in S$ – we shall show that $x_i \in U(b_0)$. Suppose that $x_i \notin U(b_0)$. Since X is compact, and hence regular, there is an open set V such that $x_i \in V$ and $V \cap U(b_0) = \emptyset$. But x_i is a cluster point of $\{x_i^{T_{12\ldots B}} : T_{12\ldots B} \in \Sigma_{12\ldots B}\}$, and hence of $\{x_i^T : T \in 2^S, T \text{ is finite}\}$, so there exists an element x_i^T such that $x_i^T \geq x_i^{\{b_0\}}$ and $x_i^T \in V$. Then $x_i^T \in Y_T \subset \bigcap_{b \in T} U(b) \subset U(b_0)$ – a contradiction.

Finally, we shall show that $P(\underline{a})$ fails in Z. Suppose, *a contrario,* that $P(\underline{a})$ holds true in Z. Let $\underline{t}$ be such that $p_j(\underline{a}, \underline{t}) \in D(1, q_j(\underline{a}, \underline{t}))$ in Z, $j \in \{1, \ldots, m\}$. Then $Z \subset U$, where

$$U = \bigcap_{j=1}^{m} [U(p_j(\underline{a}, \underline{t})) \cup U(-q_j(\underline{a}, \underline{t}))],$$

and $x_1, \ldots, x_B \in U$. Since x_i is a limit of the net $\{x_i^{T_{12\ldots B}} : T_{12\ldots B} \in \Sigma_{12\ldots B}\}$, there exists a $T_i \in \Sigma_{12\ldots B}$ such that $x_i^{T_{12\ldots B}} \in U$ for all $x_i^{T_{12\ldots B}} \geq x_i^{T_i}$, where $i \in \{1, \ldots, B\}$. Let $T_0 \in \Sigma_{12\ldots B}$ be such that $T_0 \supseteq T_i$, $i \in \{1, \ldots, B\}$. Then $x_1^{T_0}, x_2^{T_0}, \ldots, x_B^{T_0} \in U$, so $P(\underline{a})$ holds in Y_{T_0}. $\square$

We can now describe behavior of pp formulae under direct sums and group extensions:

2.2.5. Lemma. *Let (X, G) be a space of orderings. If $(X, G) = (X_1, G_1) \oplus \ldots \oplus (X_n, G_n)$, then the pp conjecture holds true in (X, G) for every pp formula if and only if it holds true in each of the spaces $(X_1, G_1), \ldots, (X_n, G_n)$ for every pp formula.*

2.2.6. Lemma. *Let (X, G) be a space of orderings. If (X, G) is a group extension of $(\overline{X}, \overline{G})$, then the pp conjecture holds true in (X, G) for every pp formula if and only if it holds true in $(\overline{X}, \overline{G})$ for every pp formula.*

The proof of the sufficient condition in Lemma 2.2.5 follows immediately from the definition of a direct sum and is given in [**Mar02**, Proposition 2.3]; the necessary condition is a consequence of Theorem 2.2.1 – clearly every direct summand of a given space of orderings is also a subspace of this space. Similarly, the sufficient condition in Lemma 2.2.6 is proven in [**Mar02**, Proposition 2.3], and the necessary one follows from Theorem 2.2.1: if (X, G) is a group extension of $(\overline{X}, \overline{G})$, we can choose a subgroup H of G such that $G = \overline{G} \times H$, and define the subspace $Y = \bigcap_{a \in H} U(a)$ – the space $(\overline{X}, \overline{G})$ can be identified with $(Y, G|_Y)$, the mapping from Y to $\overline{X}$ being just the restriction.

These two lemmas combined with the Structure Theorem give:

2.2.7. Theorem. *Let (X, G) be a space of orderings. If (X, G) has finite chain length, then the pp conjecture holds true in (X, G) for every pp formula.*

We shall also state the following theorem, of a similar nature ([**Mar02**, Proposition 2.3]):

2.2.8. Theorem. *Let (X, G) be a space of orderings. If (X, G) has stability index no greater than 1, then the pp conjecture holds true in (X, G) for every pp formula.*

In view of this and of Lemma 1.7.3, the pp conjecture holds true for the space of orderings of the field $\mathbb{R}(x)$ or, more generally, $R(x)$, where R is a real closed field.

It was natural to search for counterexamples to the pp conjecture among spaces of orderings of stability index 2. Recall (Lemma 1.7.3) that the space of orderings of the field $\mathbb{Q}(x)$, and spaces of orderings of function fields of rational conics, and the space of orderings of the field $\mathbb{R}(x, y)$ have stability index 2. The following theorem was proven in [**DicMarMir05**]:

2.2.9. Theorem. *For every pp formula the pp conjecture holds true for the space of orderings* $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$.

2.3. Product free and one-related pp formulae

In what follows we shall concentrate on the two remaining examples of spaces of orderings with stability index 2 mentioned at the end of the previous section, and construct pp formulae for which the pp conjecture fails. But first, before we get to this, we shall introduce a wide class of pp formulae, called product free and one-related, for which the pp conjecture holds true – the examples that will follow are the simplest cases of formulae which are not product free and one-related. Instead of investigating a pp formula in a fixed space of orderings, we shall rather consider it as an expression in the language L_{SG} of special groups. Let

$$P(\underline{y}) = \exists \underline{t} \bigwedge_{j=1}^{m} p_j(\underline{t}, \underline{y}) \in D(1, q_j(\underline{t}, \underline{y})),$$

where $\underline{t} = (t_1, \ldots, t_n)$, $\underline{y} = (y_1, \ldots, y_k)$ are tuples of individual variables in the language L_{SG}, and $p_j(\underline{t}, \underline{y})$, $q_j(\underline{t}, \underline{y})$ are $\pm$ products of some of the t_i's and y_l's, $i \in \{1, \ldots, n\}$, $l \in \{1, \ldots, k\}$. We shall define two conditions, **(A)** and **(B)**, as follows:

(A): The atomic formulae appearing in $P(\underline{y})$ which involve free variables $t_1, \ldots, t_n$ are expressible either as

$$t_i \in D(y, z),$$

or as

$$1 \in D(yt_i, zt_j), \text{ for } i \neq j,$$

or as

$$t_i \in D(yt_j, zt_k), \text{ for } i, j, k \text{ distinct},$$

where y, z are $\pm$ products of some parameters $y_1, \ldots, y_k$. Moreover, for each $i \in \{1, \ldots, n\}$, we allow any finite number of occurrences of atoms of the form $t_i \in D(y, z)$, but for each $i, j \in \{1, \ldots, n\}$, $i \neq j$, we allow at most one atom involving both t_i and t_j.

For the formula $P(\underline{y})$ satisfying **(A)** we build the **graph** of $P(\underline{y})$ as follows: there are n vertices $1, \ldots, n$, corresponding to the free variables $t_1, \ldots, t_n$, and two vertices are joined by an edge if and only if there is an atom (at most one) of $P(\underline{y})$ involving both t_i and t_j. We now introduce the second requirement:

> **(B):** The graph of $P(\underline{y})$ contains no cycles other than the 3-cycles arising from atoms of the form $t_i \in D(yt_j, zt_k)$.

A pp formula satisfying **(A)** and **(B)** will be called **product free and one-related.**

For example, consider the formula "$z \in D(y_1, \ldots, y_k)$". It is expressible in the form:

$$\exists t_1, \ldots, t_{k-2}[z \in D(y_1, t_1) \wedge t_1 \in D(y_2, t_2) \wedge \ldots \wedge t_{k-2} \in D(y_{k-1}, y_k)],$$

so we can readily see that it satisfies **(A)** and **(B)**, and its graph is:

$$\overset{1}{\bullet} \;\text{———}\; \overset{2}{\bullet} \;\text{———}\; \cdots \;\text{———}\; \overset{k-2}{\bullet}$$

Similarly, the formula "$(y_1, \ldots, y_k)$ is isotropic", equivalent to "$-y_1 \in D(y_2, \ldots, y_k)$", or:

$$\exists t_1, \ldots, t_{k-3}[-y_1 \in D(y_2, t_1) \wedge t_1 \in D(y_3, t_2) \wedge \ldots \wedge t_{k-3} \in D(y_{k-1}, y_k)],$$

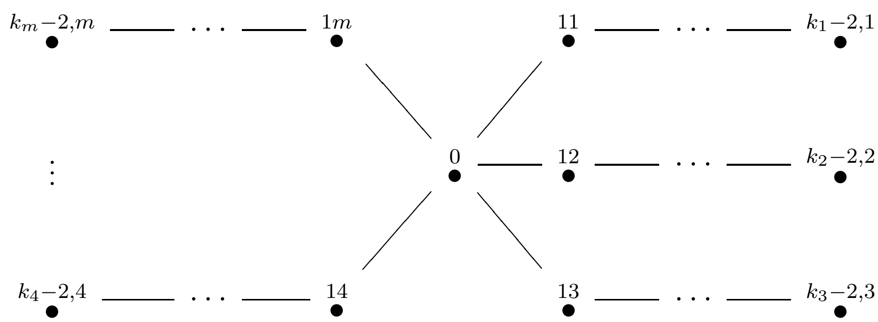
satisfies both **(A)** and **(B)**, and its graph is the following one:

$$\overset{1}{\bullet} \;\text{———}\; \overset{2}{\bullet} \;\text{———}\; \cdots \;\text{———}\; \overset{k-3}{\bullet}$$

As another example, consider the formula "$\exists t \bigwedge_{j=1}^{m} t \in D(\phi_j)$". If $\phi_j = (y_{1j}, \ldots, y_{k_j j})$, $j \in \{1, \ldots, m\}$, it can be written as:

$$\begin{aligned}
&\exists t_0, t_{11}, \ldots, t_{k_1-2,1}, t_{12}, \ldots, t_{k_2-2,2}, \ldots, t_{1m}, \ldots, t_{k_m-2,m} \\
&\qquad [t_0 \in D(y_{11}, t_{11}) \wedge t_0 \in D(y_{12}, t_{12}) \wedge \ldots \wedge t_0 \in D(y_{1m}, t_{1m}) \\
&\wedge \quad t_{11} \in D(y_{21}, t_{21}) \wedge \ldots \wedge t_{k_1-2,1} \in D(y_{k_1-1,1}, y_{k_1,1}) \\
&\ldots \\
&\wedge \quad t_{1m} \in D(y_{2m}, t_{2m}) \wedge \ldots \wedge t_{k_m-2,m} \in D(y_{k_m-1,m}, y_{k_m,m})]
\end{aligned}$$

Again, it satisfies **(A)** and **(B)**, and its graph is:



Clearly, every formula with just one quantified variable satisfies **(A)** and **(B)**, and its graph is simply a single vertex. In general, numerous properties of quadratic forms can be expressed in terms of product free and one-related pp formulae, although it is not always evident why a given formula is logically equivalent to some pp formula satisfying **(A)** and **(B).** For example, the formula "two forms are isometric" can be expressed as a product free and one-related pp formula, however the argument showing how this can be done is rather nontrivial and will not be presented here. Because of the fact that formulae logically equivalent to product free and one-related formulae seem to amount for a large percentage of pp formulae, the following result is, in fact, a very powerful theorem:

2.3.1. THEOREM. *(*[**Mar06**, Theorem 2.1]*) Let (X, G) be a space of orderings of finite stability index. Then, for any product free and one-related formula with parameters in G, the pp conjecture holds true.*

2.4. Products of value sets of binary forms

In this section we shall investigate some simple examples of pp formulae, which do not appear to be product free and one-related, and, in fact, are not, which will become evident in the next chapter. Let (X, G) be a space of orderings. Consider the statement

$$d \in \prod_{i=1}^{n} D(1, a_i),$$

where $a_1, \dots, a_n, d \in G$. Clearly, it is expressible as the following pp formula:

$$\exists t_1, \dots, t_{n-1}[t_1 \in D(1, a_1) \wedge \dots \wedge t_{n-1} \in D(1, a_{n-1}) \wedge dt_1 \dots t_{n-1} \in D(1, a_n)].$$

We would like to learn if the pp conjecture holds for this particular type of formula. We can readily see that, unless $n = 1$ or $n = 2$, it is not clear at all if this formula is logically equivalent to a product free and one-related formula, and thus, in general, Theorem 2.3.1 cannot be used here. This obstacle can be circumvented if $\mathrm{stab}(X, G) = 1$, as in this case the pp conjecture is valid for every pp formula, however we can also use another method; for every space (X, G), the set $\prod_{i=1}^{n} D(1, a_i)$ is clearly a subgroup of the value set of the Pfister form $((a_1, \ldots, a_n))$, and if $\mathrm{stab}(X, G) \leq 1$ then these two sets are, in fact, equal:

2.4.1. Lemma. *(*[**Mar06**, Proposition 3.2]*) Let (X, G) be a space of orderings of stability index at most 1, let $a_1, \ldots, a_n \in G$. Then:*

$$\prod_{i=1}^{n} D(1, a_i) = D[((a_1, \ldots, a_n))].$$

Proof. Fix $d \in D[((a_1, \ldots, a_n))]$. By Lemma 2.2.3, $\bigcap_{i=1}^{n} U(a_i) \subset U(d)$. Thus we can define continuous functions $t_1, \ldots, t_n : X \to \{-1, 1\}$ as follows:

$$\begin{aligned} t_1(x) &= \begin{cases} d(x), & \text{if } x \in U(-a_1) \\ 1, & \text{if } x \in U(a_1) \end{cases}, \\ t_2(x) &= \begin{cases} d(x), & \text{if } x \in U(a_1) \cap U(-a_2) \\ 1, & \text{if } x \in U(-a_1) \cup U(a_2) \end{cases}, \\ &\vdots \\ t_n(x) &= \begin{cases} d(x), & \text{if } x \in U(a_1) \cap \ldots \cap U(a_{n-1}) \cap U(-a_n) \\ 1, & \text{if } x \in U(-a_1) \cup \ldots \cup U(-a_{n-1}) \cup U(a_n), \end{cases} \end{aligned}$$

Since $\mathrm{stab}(X, G) \leq 1$, the strong approximation property is satisfied, and hence $t_1, \ldots, t_n \in G$. Clearly $t_1 \in D(1, a_1) \wedge t_2 \in D(1, a_2) \wedge \ldots \wedge t_n \in D(1, a_n)$, and $d = t_1 \ldots t_n$. $\square$

Now, if $\prod_{i=1}^{n} D(1, a_i) = D[((a_1, \ldots, a_n))]$, then the statement $d \in \prod_{i=1}^{n} D(1, a_i)$ is equivalent to the statement $d \in D[((a_1, \ldots, a_n))]$, and we can use the Isotropy Theorem to decide whether the pp conjecture is valid or not.

Unfortunately, we cannot use this method when $\mathrm{stab}(X, G) \geq 2$ – in this case the equality $\prod_{i=1}^{n} D(1, a_i) = D[((a_1, \ldots, a_n))]$ is no longer true. For example, suppose that (X, G) is a nontrivial fan. Let $a \in G$, $a \neq \pm 1$. Then, by Lemma 1.6.1, $D(1,a)D(1-a) = \{1,a\}\{1,-a\} = \{1,a,-a,-1\}$. But $(1,a) \otimes (1,-a) \cong (1,-1) \otimes (1,a)$, so $D(1,a) \otimes (1,-a) = G$.

Since $\prod_{i=1}^{n} D(1,a_i) = D[((a_1,\ldots,a_n))]$ can fail so easily, we shall replace the product $\prod_{i=1}^{n} D(1,a_i)$ with $\prod_{\delta\in\{0,1\}^n} D(1,a^\delta)$, where $a^\delta = a_1^{\delta_1}\ldots a_n^{\delta_n}$, $\delta = (\delta_1,\ldots,\delta_n)$. The equality $\prod_{\delta\in\{0,1\}^n} D(1,a^\delta) = D[((a_1,\ldots,a_n))]$ holds for fans:

2.4.2. Lemma. *(*[**Mar06**, Proposition 3.4]*) Let (X,G) be a fan, let $a_1,\ldots,a_n \in G$. Then $\prod_{\delta\in\{0,1\}^n} D(1,a^\delta) = D[((a_1,\ldots,a_n))]$.*

Proof. If, for some $\delta \in \{0,1\}^n$, $a^\delta = -1$, then there is nothing to prove. Otherwise, by Lemma 1.6.1 (2), for every $\delta \in \{0,1\}^n$, $D(1,a^\delta) = \{1,a^\delta\}$, so $\prod_{\delta\in\{0,1\}^n} D(1,a^\delta) = \{a^\delta : \delta \in \{0,1\}^n\}$. Moreover, by Lemma 1.6.1 (3), $D[((a_1,\ldots,a_n))] = \{a^\delta : \delta \in \{0,1\}\}$. □

The conclusion of the above lemma can fail for spaces which are not fans, which are group extensions of fans, and whose stability index is 2, for $n \geq 2$. To be more specific, we have:

2.4.3. Lemma. *(*[**Mar06**, Proposition 3.5]*) Let (X,G) be a space of orderings which is not a fan, and let* $\mathrm{stab}(X,G) = 2$. *Let (X,G) be a proper group extension of $(\overline{X},\overline{G})$. If $a_1,\ldots,a_n \in \overline{G}$, and if $a^\delta \neq -1$ for every $\delta \in \{0,1\}^n$, and if $((a_1,\ldots,a_n))$ is isotropic, then*

$$\prod_{\delta\in\{0,1\}^n} D(1,a^\delta) = \overline{G} \text{ and } D[((a_1,\ldots,a_n))] = G.$$

Otherwise, $\prod_{\delta\in\{0,1\}^n} D(1,a^\delta) = D[((a_1,\ldots,a_n))]$.

This follows from the description of value sets in group extensions, the fact that $\mathrm{stab}(\overline{X},\overline{G}) \leq 1$, and Lemma 2.4.1. If (X,G) is a finite space, it is, by the Structure Theorem, a direct sum of its connected components, and the above lemma can be rephrased as follows:

2.4.4. Lemma. *(*[**Mar06**, Proposition 3.6]*) Let (X,G) be a finite space of orderings, and let* $\mathrm{stab}(X,G) = 2$. *Then $d \in \prod_{\delta\in\{0,1\}^n} D(1,a^\delta)$ if and only if, for each connected component (Y,H) of (X,G) which is not a fan, which is a proper group extension of some space, and whose residue space is $(\overline{Y},\overline{H})$, if images of $a_1,\ldots,a_n$ in H are in $\overline{H}$, and if images of a^δ in H are not -1 for all $\delta \in \{0,1\}^n$, and if $((a_1,\ldots,a_n))$ is isotropic in (Y,H), then the image of d in H belongs to $\overline{H}$.*

CHAPTER 3

Spaces of orderings of rational conics

This chapter is an expanded version of our work [**GłaMar-1**]: we classify spaces of orderings of function fields of rational conic sections with respect to the pp conjecture. In what follows the only "nontrivial" cases are the ones of an ellipse without rational points, and two parallel lines without rational points. Preliminary results stated in Section 1 are taken from [**DicMarMir05**]; we make an extensive use of the Tarski Transfer Principle – the reader might wish to consult, for example, the monograph [**BCR**]. The part of Section 1 which deals with prime cones and orderings of rings is also drawn from [**BCR**]. Sections 2, 3 and 4 is the main part of [**GłaMar-1**]; Theorem 3.2.6 is an old result proven, for example, in [**Sam61**], however here an elementary proof is given. In Section 5 we use the fact that subspaces of the space $(X_{\mathbb{Q}(x_1,\ldots,x_n)}, G_{\mathbb{Q}(x_1,\ldots,x_n)})$ compatible to certain valuations are group extensions of spaces of orderings of rational conics to disprove the pp conjecture in $(X_{\mathbb{Q}(x_1,\ldots,x_n)}, G_{\mathbb{Q}(x_1,\ldots,x_n)})$.

3.1. Spaces of orderings of function fields

Let K be a uniquely ordered field. Let $\mathfrak{p}$ be a prime ideal of the ring $K[x_1,\ldots,x_n]$, and consider the field $F = (K[x_1,\ldots,x_n]/\mathfrak{p})$. We assume that F is formally real. With a slight abuse of notation we shall use the same symbols to denote elements of $K[x_1,\ldots,x_n]$ and elements of $K[x_1,\ldots,x_n]/\mathfrak{p}$.

Let R be the real closure of K, that is an algebraic extension of K extending the ordering of K, and maximal with respect to that property. Let V denote the zero set of $\mathfrak{p}$ in R^n:

$$V = \{(a_1,\ldots,a_n) \in R^n : \forall f \in \mathfrak{p}[f(a_1,\ldots,a_n) = 0]\}.$$

In general, we shall denote by $\mathcal{Z}(I)$ the zero set of an ideal I.

We need to give a geometric meaning to the formula $f \in D(g,h)$ in a subspace X_T of (X_F, G_F), where T is a finitely generated preordering. In order to do that, we will need a generalization of the notion of orderings to the ring case (see Chapter 7 of [**BCR**]). Let A be a commutative ring with identity. A subset $\alpha \subsetneq A$ is called a **prime cone** if $\alpha + \alpha \subset \alpha$, $\alpha \cdot \alpha \subset \alpha$, $A^2 \subset \alpha$, and, for $ab \in \alpha$, either $a \in \alpha$ or $-b \in \alpha$. For a fixed prime cone α the set $\alpha \cap -\alpha$ shall be denoted by $\mathrm{supp}\alpha$ and called the **support** of α. It can be easily checked that the support of a prime cone is a prime ideal.

We start with the following lemma ([**BCR**, Proposition 4.3.4]):

3.1.1. Lemma. *Let A be a commutative ring. A subset $\alpha \subset A$ is a prime cone of A if and only if there exist a formally real field $(L, \leq)$ and a homomorphism $\phi : A \to L$ such that $\alpha = \{a \in A : \phi(a) \geq 0\}$.*

Proof. Given a formally real field $(L, \leq)$ and a homomorphism $\phi : A \to L$, one easily checks that $\{a \in A : \phi(a) \geq 0\}$ is a prime cone. Conversely, if α is a prime cone of A, then, since $\mathrm{supp}\alpha$ is a prime ideal, $A/\mathrm{supp}\alpha$ is a domain, which can be endowed with a prime cone P defined as follows:

$$a + \mathrm{supp}\alpha \in P \Leftrightarrow a \in \alpha.$$

Denote by κ the canonical epimorphism $A \to A/\mathrm{supp}\alpha$. We define an ordering $\overline{P}$ of the field of fractions $(A/\mathrm{supp}\alpha)$ of the domain $A/\mathrm{supp}\alpha$ in the following manner:

$$\frac{a + \mathrm{supp}\alpha}{b + \mathrm{supp}\alpha} \in \overline{P} \Leftrightarrow (a + \mathrm{supp}\alpha)(b + \mathrm{supp}\alpha) \in P,$$

and we denote by q the canonical embedding $A/\mathrm{supp}\alpha \hookrightarrow (A/\mathrm{supp}\alpha)$. Finally, let $\phi = q \circ \kappa$. It is easy to check that $\overline{P}$ is closed under addition and multiplication, and that $\overline{P} \cup -\overline{P} = (A/\mathrm{supp}\alpha)$. We shall show that $\alpha = \phi^{-1}(\overline{P})$, which will also imply that $\overline{P} \cap -\overline{P} = \{0\}$.

Let $a + \mathrm{supp}\alpha \in \mathrm{Im}\phi$ and suppose that $a + \mathrm{supp}\alpha \in \overline{P}$. Then $a + \mathrm{supp}\alpha = \frac{b+\mathrm{supp}\alpha}{c+\mathrm{supp}\alpha}$ for some $b, c \in A$, with $c \notin \mathrm{supp}\alpha$ and $bc \in \alpha$. We may assume that $c \in \alpha$. Then, since $-c \notin \alpha$, we have that $b \in \alpha$. Moreover, $ac = b + d$ for some $d \in \mathrm{supp}\alpha$, and thus $ac \in \alpha$. Again, since $-c \notin \alpha$, it follows that $a \in \alpha$. $\square$

We see that the support of a prime cone is always a **real prime ideal**, that is a prime ideal I such that if $a_1^2 + \ldots + a_p^2 \in I$, where $a_1, \ldots, a_p \in A$, then $a_1, \ldots, a_p \in I$.

In the next lemma we shall describe how a prime cone of $K[x_1, \ldots, x_n]$ can be extended to a prime cone of $R[x_1, \ldots, x_n]$. We will write $\underline{x}$ for $(x_1, \ldots, x_n)$.

3.1.2. Lemma. *Let K and R be defined as above. Every prime cone α of $K[\underline{x}]$ extends uniquely to a prime cone β of $R[\underline{x}]$ such that $\alpha = K[\underline{x}] \cap \beta$.*

Proof. Let α be a prime cone of $K[\underline{x}]$. By the previous lemma, there is a formally real field $(L, \leq)$ and a homomorphism $\phi : K[\underline{x}] \to L$ such that $\alpha = \{f \in K[\underline{x}] : \phi(f) \geq 0\}$. We may assume that L is real closed. By [**BCR**, Proposition 1.3.4], there is a unique order preserving homomorphism $\psi : R \to L$ such that $\psi|_K = \phi|_K$. This homomorphism extends in an obvious way to a unique homomorphism $\tilde{\psi} : R[\underline{x}] \to L$ such that $\tilde{\psi}|_{K[\underline{x}]} = \phi$ which, by the previous lemma, corresponds to a prime cone β of $R[\underline{x}]$. Obviously $K[\underline{x}] \cap \beta = \alpha$. $\square$

We now proceed to the main lemma:

3.1.3. Lemma. *([**DicMarMir05**, Theorem 3.1]) Let K, F and V be defined as before. For $f, g_1, \ldots, g_s$ non-zero elements of $K[\underline{x}]/\mathfrak{p}$, the condition*

$$\forall_{P \in X_F} (g_1, \ldots, g_s \in P \Rightarrow f \in P)$$

holds true if and only if, for every irreducible component W of V of maximal dimension, and for every regular point $a \in W$:

$$g_1(a) > 0, \ldots, g_s(a) > 0 \Rightarrow f(a) \geq 0.$$

Proof. $(\Leftarrow)$. Let $P \in X_F$ and define a prime cone α in $K[\underline{x}]$ by

$$f \in \alpha \Leftrightarrow f + \mathfrak{p} \in P.$$

Then $\mathfrak{p} = \operatorname{supp}\alpha$. Let β be the unique prime cone in $R[\underline{x}]$ such that $\alpha = K[\underline{x}] \cap \beta$, and denote by $\mathfrak{q}$ the support of β. Consider the zero set $W = \mathcal{Z}(\mathfrak{q})$ of $\mathfrak{q}$ in R^n; clearly, W is irreducible and $W \subset V$. Observe that W is also a component of V of maximal dimension, that is $\dim W = \dim V$. Indeed, $R[\underline{x}]$ is integral over $K[\underline{x}]$, and thus $R[\underline{x}]/\mathfrak{q}$ is integral over $K[\underline{x}]/\mathfrak{p}$. Moreover, $\mathfrak{q}$ is a real prime ideal, so $\dim W = \dim R[\underline{x}]/\mathfrak{q}$, and, in turn, $\dim W = \dim K[\underline{x}]/\mathfrak{p}$

(see [**AM69**, Corollary 5.9 and Theorem 5.11]). On the other hand, $\mathfrak{p} \subset K[\underline{x}] \cap \mathcal{I}(V) \subset K[\underline{x}] \cap \mathfrak{q}$, which implies $\mathfrak{p} = K[\underline{x}] \cap \mathcal{I}(V)$. Henceforth $\dim V = \dim K[\underline{x}]/\mathfrak{p} = \dim W$.

Let $\mathfrak{q} = (f_1, \ldots, f_r)$ for $f_1, \ldots, f_r \in R[\underline{x}]$. The statement

"a is a regular point of W such that if $g_1(a) > 0, \ldots, g_s(a) > 0$ then $f(a) \geq 0$"

is expressible as the following formula in the field R:

$$f_i(a) = 0, i \in \{1, \ldots, r\} \wedge r\left[\frac{\partial f_i}{\partial x_j}(a)\right]_{\substack{i\in\{1,\ldots,r\},\\ j\in\{1,\ldots,n\}}} = n - \dim W \wedge g_k(a) > 0, k \in \{1, \ldots, s\}$$
$$\Rightarrow f(a) \geq 0,$$

where $r(M)$ denotes the rank of a matrix M. By the Tarski Transfer Principle, this implies that the following formula holds in the real closure of the field $(R[\underline{x}]/\mathfrak{q})$:

$$f_i(\beta) = 0, i \in \{1, \ldots, r\} \wedge r\left[\frac{\partial f_i}{\partial x_j}(\beta)\right]_{\substack{i\in\{1,\ldots,r\},\\ j\in\{1,\ldots,n\}}} = n - \dim W \wedge g_k(\beta) > 0, k \in \{1, \ldots, s\}$$
$$\Rightarrow f(\beta) \geq 0,$$

where $h(\beta)$ denotes the image of $h + \mathfrak{q}$ under the natural embedding of $R[\underline{x}]/\mathfrak{q}$ into the real closure of $(R[\underline{x}]/\mathfrak{q})$. The first part of the formula, that is $f_i(\beta) = 0$, for $i \in \{1, \ldots, r\}$, and $r\left[\frac{\partial f_i}{\partial x_j}(\beta)\right]_{i\in\{1,\ldots,r\},j\in\{1,\ldots,n\}} = n - \dim W$, is trivially satisfied; $g_k(\beta) > 0$ obviously implies $g_k(\alpha) > 0$, which, in turn, translates as $g_k \in P$, $k \in \{1, \ldots, s\}$ – similarly for $f(\beta) \geq 0$.

($\Rightarrow$) Each irreducible component W of V corresponds to a real prime ideal $\mathfrak{q}$ of $R[\underline{x}]$ such that $\mathfrak{q} \cap K[\underline{x}] \supset \mathfrak{p}$. We have already seen that $\dim V = \dim K[\underline{x}]/\mathfrak{p}$. If $\mathfrak{q} \cap K[\underline{x}] = \mathfrak{p}$, then $\dim W = \dim K[\underline{x}]/\mathfrak{p} = \dim V$. Conversely, if $\mathfrak{q} \cap K[\underline{x}] \supsetneq \mathfrak{p}$, then a chain of prime ideals of $K[\underline{x}]/\mathfrak{q} \cap K[\underline{x}]$ of maximal length would give a rise to a longer chain of prime ideals in $K[\underline{x}]/\mathfrak{p}$, and thus $\dim W = \dim K[\underline{x}]/\mathfrak{q} \cap K[\underline{x}] < \dim K[\underline{x}]/\mathfrak{p} = \dim V$. Therefore W has maximal dimension if and only if $\mathfrak{q} \cap K[\underline{x}] = \mathfrak{p}$.

Let W be an irreducible component of V of maximal dimension, let $\mathfrak{q} = \mathcal{I}(W)$ be the ideal associated to W in $R[\underline{x}]$, and let $a \in W$ be a regular point of W. Consider the prime cone $\beta_a = \{h + \mathfrak{q} : h(a) \geq 0\}$ of $R[\underline{x}]/\mathfrak{q}$. By [**BCR**, Proposition 7.6.2], there is a prime cone

β of $R[\underline{x}]$ which induces a prime cone of $R[\underline{x}]/\mathfrak{q}$ contained in β_a, and such that $\mathrm{supp}\beta = \mathfrak{q}$. β restricts to a prime cone α of $K[\underline{x}]$, and since W is of maximal dimension, $\mathrm{supp}\alpha = \mathfrak{p}$. If $g_1(a) > 0, \dots, g_s(a) > 0$, then $g_1(\alpha) > 0, \dots, g_s(\alpha) > 0$, so that $f(\alpha) > 0$ and, consequently, $f(a) \geq 0$. $\square$

This lemma combined with Lemma 2.2.3 gives:

3.1.4. Theorem. *([**DicMarMir05**, Corollary 3.2])Let K and F be defined as before, let $f, g, h, g_1, \dots, g_s$ be non-zero elements of the ring $K[\underline{x}]/\mathfrak{p}$. Let T be a preordering of F generated by $g_1, \dots, g_s$. Then $f|_{X_T} \in D_{X_T}(g|_{X_T}, h|_{X_T})$ if and only if, for every irreducible component W of V of maximal dimension, and for every regular point $a \in W$, if $g_1(a) > 0, \dots, g_s(a) > 0$ then $f(a)g(a) \geq 0$ or $f(a)h(a) \geq 0$.*

3.2. Coordinate rings and function fields of conics

We shall classify function fields of rational conics with respect to the pp conjecture. Everything we do to begin with works equally well with $\mathbb{Q}$ replaced by any field K, $\mathrm{char}K \neq 2$, however we will not pursue this more general setting any further here because it is somewhat tangential to our discussion. If a function field of a rational conic is purely transcendental over $\mathbb{Q}$, then Theorem 2.2.9 shows that the pp conjecture holds for every pp formula, so we focus our attention here on the remaining cases. We start clasifying function fields of conics with the following well known fact:

3.2.1. Lemma. *Let $f \in \mathbb{Q}[x, y]$ be a polynomial of degree 2. The curve*

(1) $$\mathcal{C} : f(x, y) = 0,$$

is affine isomorphic either to a curve of parabolic type:

(2) $$ax^2 + y = 0, \quad a \in \mathbb{Q}^*,$$

or to a curve of parallel type:

(3) $$ax^2 + c = 0, \quad a \in \mathbb{Q}^*, c \in \mathbb{Q},$$

or to a curve of elliptic (hyperbolic) type:

$$ax^2 + by^2 + c = 0, \quad a, b \in \mathbb{Q}^*, c \in \mathbb{Q}. \tag{4}$$

3.2.2. Lemma. *If the irreducible curve (1) is affine isomorphic to the parabolic curve (2), then its function field $\mathbb{Q}(\mathcal{C})$ is a purely transcendental extension of $\mathbb{Q}$ of degree 1.*

Proof. If $\mathbb{Q}(\mathcal{C}) \cong \mathbb{Q}(x, y)$, where x, y are elements transcendental over $\mathbb{Q}$ such that $ax^2 + y = 0$, $a \neq 0$, then $\mathbb{Q}(\mathcal{C}) \cong \mathbb{Q}(x, y) = \mathbb{Q}(x, -ax^2) = \mathbb{Q}(x)$. $\square$

Thus, the pp conjecture holds for every pp formula in a space of orderings of a function field of a curve of type (2). Next, among all non-parabolic conics we shall distinguish between curves having rational points, and curves without such points.

3.2.3. Lemma. *If the irreducible curve (1) has a rational point, then it is affine isomorphic to a curve of type either (2) or (4).*

Proof. Suppose that $\mathcal{C}$ is affine isomorphic to a curve $ax^2 + c = 0$ and has a rational point (q, r). Then

$$ax^2 + c = ax^2 - aq^2 = a(x - q)(x + q),$$

so that $\mathcal{C}$ is reducible – a contradiction. $\square$

Therefore, we shall concentrate on curves of type (3) without rational points, and curves of type (4) with or without rational points. The case of the curve (4) breaks into two subcases: consider an irreducible curve

$$ax^2 + by^2 = 0, \quad a, b \in \mathbb{Q}^*. \tag{5}$$

This curve is birationally isomorphic to the curve (3) without rational points. Indeed, the mapping $(x, y) \mapsto (\frac{x}{y}, 1)$ maps (5) onto $\mathcal{C}' : a(\frac{x}{y})^2 + b = 0$. Clearly, if (p, q) is a rational point on $\mathcal{C}'$, then $b = -ap^2$ and $ax^2 + by^2 = a(x - py)(x + py)$, so that (5) is reducible.

Thus we are down to the case of curves (3) without rational points and curves (4) with $c \neq 0$, with or without rational points. The next lemma eliminates one more case:

3.2.4. Lemma. *If the irreducible curve (1) is affine isomorphic to*

$$ax^2 + by^2 + c = 0, \qquad a, b, c \in \mathbb{Q}^*. \tag{6}$$

and has a rational point, then $\mathbb{Q}(\mathcal{C}) \cong \mathbb{Q}(z)$ *for a* z *transcendental over* $\mathbb{Q}$.

Proof. By assumption, $\mathbb{Q}(\mathcal{C}) \cong \mathbb{Q}(x, y)$, where $ax^2+by^2+c = 0$ and x, y are transcendental over $\mathbb{Q}$. Moreover, $aq^2+br^2+c = 0$ for some rational point (q, r). Thus $ax^2-aq^2 = br^2-by^2$. Let $z = \frac{x-q}{y-r}$. Hence $\mathbb{Q}(z) \subset \mathbb{Q}(x, y)$. Conversely, we have:

$$az(x + q) = a\frac{x-q}{y-r}(x + q) = \frac{ax^2 - aq^2}{y-r} = -\frac{by^2 - br^2}{y-r} = -b(y + r),$$

and after rearranging:

$$azx + by = -azq - br. \tag{7}$$

On the other hand, the equation $z = \frac{x-q}{y-r}$ gives:

$$x - zy = q - zr. \tag{8}$$

The determinant $-az^2 - b$ of the system of equations (7) and (8) is nonzero; if it was zero, then $a(x - q)^2 + b(y - r)^2 = 0$. This implies that the irreducible polynomial $ax^2 + by^2 + c$ divides the polynomial $a(x-q)^2 + b(y-r)^2$ so, comparing coefficients, they are equal. Then, comparing coefficients some more, $q = r = 0$ and $c = 0$, which contradicts $c \neq 0$. □

In particular, the pp conjecture holds for every pp formula in a space of orderings of a function field of a curve of type (6) with a rational point.

The remaining two cases – curves of type (3) or (6) without rational points – are more complicated. We shall show that in each of these two cases the pp conjecture fails for some pp formula. In order to do that, we need a better understanding of orderings of function fields of such curves. Fortunately enough, coordinate rings of considered curves are PID, which allows us to give a complete description of valuations of their function fields. We shall discuss this now in some more detail. For a curve $\mathcal{C}$ affine isomorphic to (3) we have $\mathbb{Q}[\mathcal{C}] \cong \mathbb{Q}[x, y]$, where x and y are transcendental over $\mathbb{Q}$ and $ax^2 + c = 0$, so that $\mathbb{Q}[\mathcal{C}] \cong \mathbb{Q}[\sqrt{-\frac{c}{a}}][y] = \mathbb{Q}(\sqrt{-\frac{c}{a}})[y]$

is a PID. The case of a curve of the type (6) requires more work; we start with the following well known result:

3.2.5. Lemma. *Let R be a PID, let $\Delta \in R$ be a square-free element, and let $2 \in R^*$. Then $R[\sqrt{\Delta}]$ is a Dedekind domain.*

Proof. Since the integral closure of a Dedekind domain in a finite extension of its quotient field is a Dedekind domain, it suffices to show that $R[\sqrt{\Delta}]$ is the integral closure of R. An element $\alpha + \beta\sqrt{\Delta} \in R[\sqrt{\Delta}]$ is a root of the polynomial $T^2 - 2\alpha T + \alpha^2 - \Delta\beta^2$ in $R[T]$, and hence is integral over R. Conversely, fix an element $g = \alpha + \beta\sqrt{\Delta} \in (R[\sqrt{\Delta}])$ integral over R, $\alpha, \beta \in (R)$. Since the mapping:

$$(R[\sqrt{\Delta}]) \ni \varphi + \psi\sqrt{\Delta} = h \mapsto \overline{h} = \varphi - \psi\sqrt{\Delta} \in (R[\sqrt{\Delta}])$$

is an R-automorphism, $\overline{g}$ is integral and so is $g + \overline{g} = 2\alpha$. Since 2 is invertible in R, also α is integral, which means that $\alpha \in R$.

Now $\beta\sqrt{\Delta} = g - \alpha$ is integral and, consequently, $\beta^2\Delta$ is integral. But $\beta^2\Delta \in (R)$, implying that $\beta^2\Delta \in R$. Let $\varphi, \psi \in R$ be two elements with no common factors and such that $\beta = \frac{\varphi}{\psi}$. Then $(\frac{\varphi}{\psi})^2\Delta = \eta$ for some $\eta \in R$, which gives $\varphi^2\Delta = \psi^2\eta$. But Δ is square-free and φ, ψ have no common divisor except for a unit, so, since R is a UFD, ψ^2 has to be a unit in R and, consequently, ψ is also a unit. Therefore $\beta = \frac{\varphi}{\psi} \in R$ proving that $g \in R[\sqrt{\Delta}]$. $\square$

This lemma applies to the coordinate ring of the irreducible curve $\mathcal{C}$ of type (6) with no rational points: since $c \neq 0$, $\mathbb{Q}[\mathcal{C}] \cong R[\sqrt{\Delta}]$, where $R = \mathbb{Q}[x]$ and $\Delta = -\frac{c}{b} - \frac{a}{b}x^2$. Observe that Δ is irreducible in R, and hence square-free (if Δ was reducible, then $\mathcal{C}$ would have rational points).

3.2.6. Theorem. *([**GłaMar-1**, Lemma 1]) The coordinate ring $\mathbb{Q}[\mathcal{C}]$ of the irreducible curve (3) or (6) with no rational points is a PID.*

Proof. We have already discussed type (3) curves. Let $\mathbb{Q}[\mathcal{C}] \cong R[\sqrt{\Delta}]$, with $R = \mathbb{Q}[x]$ and $\Delta = -\frac{c}{b} - \frac{a}{b}x^2$ square free. It suffices to show that every prime ideal $\mathfrak{P}$ in $R[\sqrt{\Delta}]$ is principal. Indeed, $\mathfrak{P} \cap R = (\pi)$ for some prime $\pi \in R$. The extended ideal $(\pi)^e$ in $R[\sqrt{\Delta}]$ is just the principal ideal generated in $R[\sqrt{\Delta}]$ by π. There are two possible decompositions of $(\pi)^e$ into a product of prime ideals ([**ZarSam58**, Theorem V.13, Theorem V.22]):

$$(\pi)^e \text{ is prime and } [R[\sqrt{\Delta}]/(\pi)^e : R/(\pi)] = 2, \tag{9}$$

$$(\pi)^e = \mathfrak{P}\overline{\mathfrak{P}} \text{ and } [R[\sqrt{\Delta}]/\mathfrak{P} : R/(\pi)] = [R[\sqrt{\Delta}]/\overline{\mathfrak{P}} : R/(\pi)] = 1, \tag{10}$$

where $\overline{\mathfrak{P}}$ denotes the image of $\mathfrak{P}$ under the conjugate automorphism (note that $\mathfrak{P}$ and $\overline{\mathfrak{P}}$ may be equal). If (9) holds then, since $R[\sqrt{\Delta}]$ is a Dedekind domain, $(\pi)^e = \mathfrak{P}$ and there is nothing left to prove. If (10) holds then there are two cases to consider. Firstly, suppose that $\sqrt{\Delta} \in \mathfrak{P}$. Then $\Delta = (\sqrt{\Delta})^2 \in \mathfrak{P} \cap R = (\pi)$. Since Δ and π are both irreducible, this implies $(\pi) = (\Delta)$, so $(\pi)^e = (\Delta)^e = (\sqrt{\Delta})(\sqrt{\Delta})$. Now uniqueness of factorization yields $\mathfrak{P} = \overline{\mathfrak{P}} = (\sqrt{\Delta})$.

Secondly, assume that $\sqrt{\Delta} \notin \mathfrak{P}$. Then, since $[R[\sqrt{\Delta}]/\mathfrak{P} : R/(\pi)] = 1$, we have that $\alpha + \sqrt{\Delta} \in \mathfrak{P}$ for some $\alpha \in R \setminus (\pi)$. Thus $\alpha^2 - \Delta \in (\pi)^e \cap R = (\pi)$, that is $\pi|(\alpha^2 - \Delta)$. We claim that in this case $u\pi = h\overline{h}$ for some $u \in \mathbb{Q}^*$ and $h \in R[\sqrt{\Delta}]$ – this will imply that $\mathfrak{P}$ is principal, generated either by h or by $\overline{h}$.

Replacing α by the remainder of the division of α by π, we may assume that $\deg \alpha < \deg \pi$. Observe that $\deg \pi > 1$: if $\deg \pi = 1$, then, for some $q \in \mathbb{Q}$, $\pi(q) = 0$, and hence $0 = \alpha^2(q) - \Delta(q) = \alpha^2(q) + \frac{c}{b} + \frac{a}{b}q^2$, contradicting the fact that $\mathcal{C}$ has no rational points.

We proceed by induction on $\deg \pi$. If $\deg \pi = 2$, then $\deg \alpha \le 1$, and, since $\deg \Delta = 2$, $\deg(\alpha^2 - \Delta) \le 2$. By our assumption, $\pi\gamma = \alpha^2 - \Delta$ for some $\gamma \in R$; since Δ is square-free, $\gamma \neq 0$. But $\deg \gamma = 0$, and thus $\gamma = u \in \mathbb{Q}^*$. Now simply take $h = \alpha + \sqrt{\Delta}$.

Assume that $\deg \pi > 2$ and $\gamma\pi = \alpha^2 - \Delta$ for some $\gamma \in R$. If γ is constant then, since Δ is square free, $\gamma \neq 0$, and we take $h = \alpha + \sqrt{\Delta}$. Otherwise let $\gamma = \pi_1 \cdot \ldots \cdot \pi_s$, $\pi_1, \ldots, \pi_s \in R$, be the factorization of γ into primes. Fix an arbitrary $i \in \{1, \ldots, s\}$. Since $\deg \alpha < \deg \pi$, and, consequently, $\deg \gamma < \deg \pi$, it follows that $\deg \pi_i < \deg \pi$. Clearly, $\pi_i|(\alpha^2 - \Delta)$, and, replacing α with the remainder of the division of α by π_i, if necessary, we may assume that

$\deg \alpha < \deg \pi_i$. By hypothesis, $u_i\pi_i = h_i\overline{h_i}$ for some $u_i \in \mathbb{Q}^*$, $h_i \in R[\sqrt{\Delta}]$, and hence

$$u\pi \prod_{i=1}^{s} h_i\overline{h_i} = h_0\overline{h_0},$$

where $h_0 = \alpha - \sqrt{\Delta}$ and $u = u_1 \cdot \ldots \cdot u_s$.

We know that (h_i) and $(\overline{h}_i)$ are prime ideals in $R[\sqrt{\Delta}]$ and that $(\pi_i)^e = (h_i)(\overline{h}_i)$, $i \in \{1, \ldots, s\}$. $R[\sqrt{\Delta}]$ is a Dedekind domain, so for each $i \in \{1, \ldots, s\}$ either (h_i) or $(\overline{h}_i)$ appears in the prime factorization of (h_0) – we may assume the former is always the case. Thus $(h_1 \ldots h_s)$ is a factor of (h_0) and, for some $h \in R[\sqrt{\Delta}]$, $h_0 = h_1 \ldots h_s h$, so that $u\pi = h\overline{h}$. □

3.3. The pp conjecture for function fields of elliptic conics over $\mathbb{Q}$

We continue our discussion with curves of type (6) without rational points. To avoid trivial cases, we shall assume existence of some real points on such curves. Since now we need to look at these curves more "geometrically", we note that, after scaling and/or interchanging x and y (if necessary), the curve (6) clearly satisfies either:

(11) $$a > 0,\ \ b > 0,\ \ c < 0, \quad \text{(elliptic type)}, \qquad \text{or}$$

(12) $$a > 0,\ \ b < 0,\ \ c < 0, \quad \text{(hyperbolic type)}.$$

3.3.1. Lemma. *The curve (6) of type (12) without rational points is birationally isomorphic to the curve (6) satisfying (11) with no rational points.*

Proof. The birational isomorphism between the curves satisfying (12) and (11) is given by $(x, y) \mapsto (\frac{y}{x}, \frac{1}{x})$ and maps $\mathcal{C} : ax^2 + by^2 + c = 0$ onto $\mathcal{C}' : b(\frac{y}{x})^2 + c(\frac{1}{x})^2 + a = 0$. Suppose that the resulting curve has a rational point (q, r). If $r \neq 0$ then $(\frac{1}{r}, \frac{q}{r})$ is a rational point on $\mathcal{C}$ – a contradiction. If $r = 0$, then we can parameterize rational points (q', r') on $\mathcal{C}'$, for which $r' \neq 0$, by lines with rational slopes passing through $(q, 0)$; the set of such points is clearly nonempty, which again yields a contradiction. □

Therefore, we can restrict ourselves to the curves in a shape of an ellipse, that is satisfying (11). We record the following result which we need later ([**GłaMar-1**, Lemma 2]):

3.3.2. Lemma. *Let $\mathcal{C}$ be the irreducible curve (6) without rational points satisfying (11).*

(1) *The units of the coordinate ring $\mathbb{Q}[\mathcal{C}]$ are precisely the elements of $\mathbb{Q}^*$.*

(2) *Linear functions $p = rx + sy + t$, for $r, t \in \mathbb{Q}$, $s \in \mathbb{Q}^*$, are irreducible in $\mathbb{Q}[\mathcal{C}]$.*

Proof. Let $\mathbb{Q}[\mathcal{C}] \cong \mathbb{Q}[x, y] = R[\sqrt{\Delta}]$, with $R = \mathbb{Q}[x]$, $\Delta = y^2 = -\frac{c}{b} - \frac{a}{b}x^2$ square free.

(1). Let $\alpha + \beta\sqrt{\Delta}$ be a unit in $R[\sqrt{\Delta}]$. Then $\alpha^2 - \beta^2\Delta = \alpha^2 + \beta^2\frac{c}{b} + \beta^2\frac{a}{b}x^2$ is a unit in R, that is a polynomial of degree zero. If $\beta = 0$ then we are done. Suppose that $\beta \neq 0$. Comparing the leading coefficients of α^2 and $\beta^2\Delta$, we see that $-\frac{a}{b}$ is a square in $\mathbb{Q}$, which, since $a > 0$ and $b > 0$, is impossible.

(2). The product of p and its conjugate $\overline{p}$ is equal to $(rx+t)^2 - s^2\Delta = \frac{s^2}{b}[ax^2 + b(\frac{rx+t}{s})^2 + c]$, and is an irreducible polynomial in $\mathbb{Q}[x]$; if it was reducible, then it had a rational root and, consequently, $\mathcal{C}$ would have a rational point. It follows that p is irreducible in $\mathbb{Q}[\mathcal{C}]$. $\square$

Observe that, for two real points ξ^1, ξ^2 on the irreducible curve $\mathcal{C}$ without rational points satisfying (6) and (11), we may pick rational points q^1, q^2 lying arbitrarily close to ξ^1, ξ^2, and the element $p \in \mathbb{Q}[\mathcal{C}]$ describing the line passing through q^1 and q^2 is irreducible.

We shall describe orderings of the field F of rational functions of the ellipse $\mathcal{C}$ without rational points defined by (6) and (11). Let $p \in \mathbb{Q}[\mathcal{C}]$ be an irreducible element. Then p gives a rise to a valuation $v_p : F \to \mathbb{Z} \cup \{\infty\}$, which acts on regular functions as follows:

$$v_p(f) = \begin{cases} \infty & \text{if } f = 0, \\ k & \text{if } f = p^k \cdot g \text{ and } p \nmid g, g \in \mathbb{Q}[\mathcal{C}]. \end{cases} \tag{13}$$

The residue field F_{v_p} is isomorphic to $\mathbb{Q}[\mathcal{C}]/(p)$, and the number of orderings of F_{v_p} is equal to the number of real points ξ on $\mathcal{C}$ satisfying $p(\xi) = 0$. Say ξ is one of those points; it gives a rise to an Archimedean ordering of F_{v_p} and, by the Baer-Krull correspondence, to a pair of non-Archimedean orderings Q_ξ^+ and Q_ξ^- on F which, in terms of regular functions on $\mathcal{C}$, can be described as follows: if $f = p^k \cdot g$, $p \nmid g$, $f, g \in \mathbb{Q}[\mathcal{C}]$, then:

$$f \in Q_\xi^+ \Leftrightarrow g(\xi) > 0.$$

$$f \in Q_\xi^- \Leftrightarrow (g(\xi) > 0 \wedge k \text{ even}) \vee (g(\xi) < 0 \wedge k \text{ odd}),$$

Let P be an ordering of F. The valuation ring

$$B = \{f \in F : n + f, n - f \in P \text{ for some integer } n \geq 1\}$$

is either equal to F, in which case P is Archimedean, or is associated with some nontrivial valuation v. Then, since $ax^2 + by^2 + c = 0$ and $a > 0$, $b > 0$, $c < 0$, we have that

$$0 = v(-c) = v(ax^2 + by^2) = 2 \min\{v(x), v(y)\},$$

proving that $x, y \in B$, so $\mathbb{Q}[\mathcal{C}] \subset B$. Because $\mathbb{Q}[\mathcal{C}]$ is a PID, $v = v_p$ is one of the valuations (13) induced by some irreducible element $p \in \mathbb{Q}[\mathcal{C}]$, and P is one of the orderings compatible with v_p of the form described above.

3.3.3. Lemma. *(*[**GłaMar-1**, Lemma 3]*) Let p be a prime element in the coordinate ring $\mathbb{Q}[\mathcal{C}]$ of the irreducible curve (6) without rational points satisfying (11), let ξ be a real point of intersection of $p(x, y) = 0$ with $\mathcal{C}$. Then p changes sign on $\mathcal{C}$ at ξ and, counting the number of sign changes, p intersects with $\mathcal{C}$ in an even number of points.*

Proof. Let ξ be a real point on $\mathcal{C}$ such that $p(\xi) = 0$. Suppose that p does not change sign on $\mathcal{C}$ at ξ; there are two possibilities:

(1) $p(\zeta) \geq 0$ for all points ζ on $\mathcal{C}$ close to ξ,

(2) $p(\zeta) \leq 0$ for all points ζ on $\mathcal{C}$ close to ξ.

Suppose the former is the case. Observe that there is an element $u \in \mathbb{Q}[\mathcal{C}]$ such that:

$$u(\xi) > 0 \text{ and, for all points } \zeta \in \mathcal{C},\ \ u(\zeta) > 0 \Rightarrow p(\zeta) \geq 0.$$

Indeed, just take as u a polynomial with rational coefficients describing the circle centered at some rational point close to ξ, containing ξ in its interior, with rational radius sufficiently small, and signs of coefficients arranged so that $u(\zeta) > 0$ describes the inside of the circle.

By Lemma 3.1.3, if P is an ordering of the function field F of $\mathcal{C}$ such that $u \in P$, then $p \in P$. Consider the ordering Q_ξ^- induced by p. Clearly $p \notin Q_\xi^-$. Because $p(\xi) = 0$ but $u(\xi) \neq 0$, we know that $p \nmid u$. Moreover, $u(\xi) > 0$, implying that $u \in Q_\xi^-$ – a contradiction.

In (2) we proceed in a similar manner, using $-p$ instead of p and Q_ξ^+ instead of Q_ξ^-. $\square$

We believe that it is possible to prove the above lemma without use of the Tarski Transfer Principle, however we have not found a proof so far.

Next we shall state a general theorem concerning pp formulae, which is a slight modification of the respective theorem already proven for orderings of $\mathbb{Q}(x)$ stated in [**DicMarMir05**].

3.3.4. THEOREM. *(*[**GłaMar-1**, Theorem 5]*) Let $\mathcal{C}$ be the irreducible curve (6) without rational points satisfying (11), let F denote its function field. For a given pp-formula with n quantifiers $P(\underline{a})$, where $\underline{a} = (a_1, \ldots, a_k)$ are square-free parameters in $\mathbb{Q}[\mathcal{C}]$, let Σ denote the set of all irreducible factors of $a_1, \ldots, a_k$. The following conditions are equivalent:*

(1) *$P(\underline{a})$ holds true in (X_{v_p}, G_{v_p}), for every $p \in \Sigma$,*
(2) *$P(\underline{a})$ holds true in every proper subspace of (X_F, G_F),*
(3) *$P(\underline{a})$ holds true in every finite subspace of (X_F, G_F).*

PROOF. In the sequence (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) the only nontrivial part is (1) $\Rightarrow$ (2). Let $X_{\Sigma F^2[S]}$ be a proper subspace for some $S \subset F$, let $0 \neq d \in \Sigma F^2[S] \setminus \Sigma F^2$ – we may assume that $d \in \mathbb{Q}[\mathcal{C}]$. $X_{\Sigma F^2[S]} \subset X_{\Sigma F^2[d]}$, so it suffices to show that $P(\underline{a})$ holds in $(X_{\Sigma F^2[d]}, G_{\Sigma F^2[d]})$.

By Lemma 1.1.1, there exists $Q \in X_F$ such that $d \notin Q$. Thus, by the Tarski Transfer Principle, the set of real points ζ on $\mathcal{C}$ for which $d(\zeta) < 0$ is nonempty. By the continuity of d, this set is also open. Thus there exists an open arc $J \neq \emptyset$ on $\mathcal{C}$ such that $d(\zeta) < 0$ for all $\zeta \in J$. Replacing J by a possibly smaller arc, we may assume that J does not contain any of the finitely many real points of intersection of the irreducibles $p \in \Sigma$ with $\mathcal{C}$.

By Theorem 3.1.4, it suffices to show that, for any nonempty open arc J disjoint from the real points of intersection of the irreducibles $p \in \Sigma$ with $\mathcal{C}$, there exist $t_1, \ldots, t_n \in \mathbb{Q}[x, y] \setminus \{0\}$ such that, for each atom $1 \in D(a \prod_{i=1}^n t_i^{\epsilon_i}, b \prod_{i=1}^n t_i^{\delta_i})$ of the formula $P(\underline{a})$, where $\epsilon_i, \delta_i \in \{0, 1\}$, and a, b are products of $\pm$ some of a_i, and for each point $\zeta \in \mathcal{C} \setminus J$, we have:

$$a \prod_{i=1}^{n} {t_i}^{\epsilon_i}(\zeta) \geq 0 \text{ or } b \prod_{i=1}^{n} {t_i}^{\delta_i}(\zeta) \geq 0.$$

Fix such an arc J. The points of intersection of $p \in \Sigma$ with $\mathcal{C}$ divide $\mathcal{C}$ into disjoint arcs, exactly one of them containing J. Let $\mathcal{A} = \{I_1, \ldots, I_m\}$ be the set of remaining arcs. For

$I_j \in \mathcal{A}$ let $p_{I_j} \in \mathbb{Q}[\mathcal{C}]$ be a linear irreducible intersecting $\mathcal{C}$ in two points: one lying in I_j, and the other in J.

Let (Y, H) be the direct sum of the (X_{v_p}, G_{v_p}), $p \in \Sigma$. By our assumptions, $P(\underline{a})$ holds true in (Y, H). Thus there exist square-free $t_1, \ldots, t_n \in \mathbb{Q}[\mathcal{C}]$ such that, for each atom $1 \in D(a\prod_{i=1}^n t_i^{\epsilon_i}, b\prod_{i=1}^n t_i^{\delta_i})$ of the formula $P(\underline{a})$, $1 \in D(a\prod_{i=1}^n t_i^{\epsilon_i}, b\prod_{i=1}^n t_i^{\delta_i})$ holds true in (Y, H) and, consequently, in (X_{v_p}, G_{v_p}) for all $p \in \Sigma$.

Factor each t_i as $t_i = t_{i0} \cdot t_{i1}$, where t_{i0} is the product of those $p \in \Sigma$ which divide t_i, $i \in \{1, \ldots, n\}$. Fix $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$ and denote by $\overline{\overline{(\zeta^j; \zeta^{j\prime})}}$ the arc I_j with endpoints ζ^j and $\zeta^{j\prime}$ arranged in the clockwise order. Define $\mu_{ij}^- = \operatorname{sgn} t_{i1}(\zeta^j)$ and $\mu_{ij}^+ = \operatorname{sgn} t_{i1}(\zeta^{j\prime})$. Since no $p \in \Sigma$ divides t_{i1}, t_{i1} does not vanish at $\zeta^j, \zeta^{j\prime}$, and therefore $\mu_{ij}^-, \mu_{ij}^+ \in \{\pm 1\}$. Define $t_{i1}' = \pm \prod_{j=1}^m p_{I_j}^{\theta_{ij}}$ and $t_i' = t_{i0} \cdot t_{i1}'$, where the sign $\pm$ is chosen so that t_{i1}' has the same sign as t_{i1} at the ends of each of the arcs $I_j \in \mathcal{A}$, and

$$\theta_{ij} = \begin{cases} 0 & \text{if } \mu_{ij}^- = \mu_{ij}^+ \\ 1 & \text{if } \mu_{ij}^- \neq \mu_{ij}^+. \end{cases}$$

Then, for each $\zeta \in \mathcal{C} \setminus J$, $a\prod_{i=1}^n t_i'^{\epsilon_i}(\zeta) \geq 0$ or $b\prod_{i=1}^n t_i'^{\epsilon_i}(\zeta) \geq 0$. $\square$

Now we are in a position to state and prove the main theorem.

3.3.5. THEOREM. *(*[**GłaMar-1**, Theorem 6]*) Let (X_F, G_F) be a space of orderings of a function field F of an irreducible curve $\mathcal{C}$ without rational points satisfying (4) and (11). There exists a pp formula that holds in every proper subspace of (X_F, G_F) and fails in (X_F, G_F).*

PROOF. Let $p_1, \ldots, p_6 \in \mathbb{Q}[\mathcal{C}]$ be linear irreducibles which intersect with $\mathcal{C}$ as in Fig. 1. Here ξ^{1i}, ξ^{2i} denote the two real points of intersection of p_i with $\mathcal{C}$, $i \in \{1, \ldots, 6\}$, and are arranged in the above order. Replacing p_i by $-p_i$ we may assume that every p_i is positive at the origin. Let $a_1 = p_1p_6$, $a_2 = p_1p_4$, $d = -p_1p_2p_3p_5$. Consider the formula

$$P(\underline{a}) = \exists t_1 \exists t_2 \left(t_1 \in D(1, a_1) \wedge t_2 \in D(1, a_2) \wedge dt_1t_2 \in D(1, a_1a_2)\right).$$

Firstly, we shall show that $P(\underline{a})$ fails to hold in (X_F, G_F). Suppose the contrary,and let

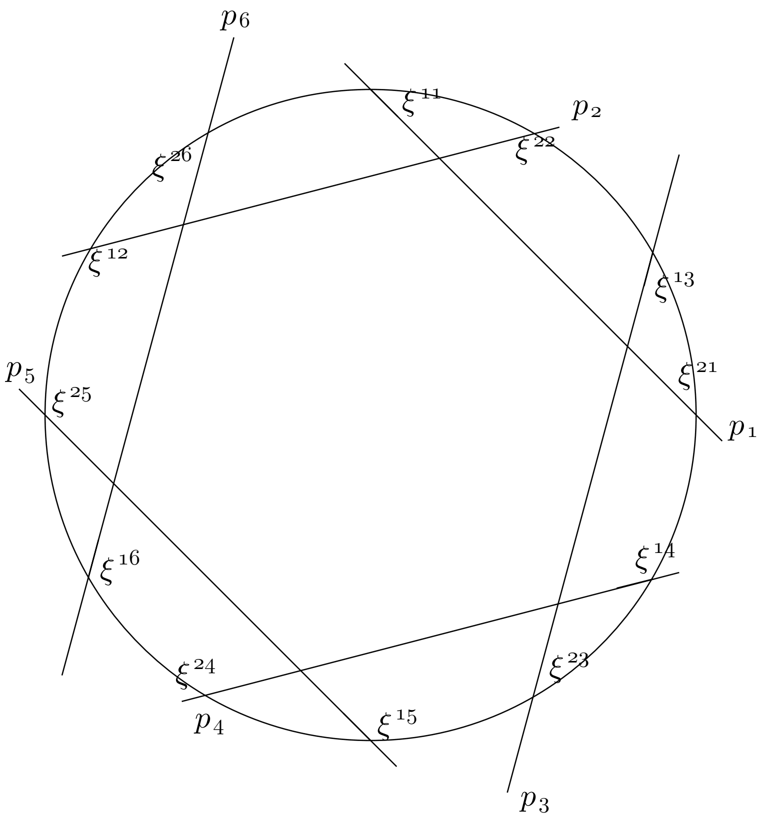


Fig. 1

$t_1, t_2 \in \mathbb{Q}[\mathcal{C}]$ be two square-free elements verifying $P(\underline{a})$. The signs of a_1, a_2 and d on the arcs between ξ^{ki}, $k \in \{1, 2\}$, $i \in \{1, \dots, 6\}$, are as follows:

	$\overline{\overline{(\xi^{11};\xi^{22})}}$	$\overline{\overline{(\xi^{22};\xi^{13})}}$	$\overline{\overline{(\xi^{13};\xi^{21})}}$	$\overline{\overline{(\xi^{21};\xi^{14})}}$	$\overline{\overline{(\xi^{14};\xi^{23})}}$	$\overline{\overline{(\xi^{23};\xi^{15})}}$	$\overline{\overline{(\xi^{15};\xi^{24})}}$	$\overline{\overline{(\xi^{24};\xi^{16})}}$	$\overline{\overline{(\xi^{16};\xi^{25})}}$	$\overline{\overline{(\xi^{25};\xi^{12})}}$	$\overline{\overline{(\xi^{12};\xi^{26})}}$	$\overline{\overline{(\xi^{26};\xi^{11})}}$
a_1	-	-	-	+	+	+	+	+	-	-	-	+
a_2	-	-	-	+	-	-	-	+	+	+	+	+
d	-	+	-	+	+	-	+	+	+	-	+	+

Observe that, for every real point ξ of $\mathcal{C}$, there is only one irreducible element of $\mathbb{Q}[\mathcal{C}]$ intersecting $\mathcal{C}$ at ξ: indeed, the kernel of the evaluation homomorphism $\mathbb{Q}[\mathcal{C}] \ni g \mapsto g(\xi) \in \mathbb{R}$ is a prime ideal generated by some irreducible element $p \in \mathbb{Q}[\mathcal{C}]$.

On the arcs $\overline{\overline{(\xi^{21};\xi^{14})}}$, $\overline{\overline{(\xi^{24};\xi^{16})}}$ and $\overline{\overline{(\xi^{26};\xi^{11})}}$ a_1 and a_2 are positive, t_1 and t_2 are nonnegative. Near ξ^{23} a_1 is positive and so is t_1. Since the only irreducible element that intersects $\mathcal{C}$ at ξ^{23} is p_3, $v_{p_3}(t_1)$ is even and t_1 does not change sign at ξ^{13}. Near ξ^{13} a_1a_2 is positive, so

dt_1t_2 is positive and d changes sign, so that t_2 changes sign. Thus $v_{p_3}(t_2)$ is odd and hence t_2 changes sign at ξ^{23}. To sum up: t_2 changes sign at ξ^{23} and ξ^{13}, but t_1 does not.

Near ξ^{12} a_1 is positive and so is t_2. Thus t_2 does not change sign at ξ^{22}. Near ξ^{22} a_1a_2 is positive and so is dt_1t_2, and d changes sign, so t_1 must change sign. Thus t_1 changes sign at ξ^{12} and ξ^{22}, but t_2 does not. Near ξ^{11} a_1a_2 is positive and so is dt_1t_2. d changes sign and so does t_1t_2. Thus one of t_1 and t_2 changes sign, but not both. Thus at ξ^{11} and ξ^{21} either t_1 changes sign (at both points), or t_2 changes sign, but not both.

On the arc $\overline{\overline{(\xi^{11};\xi^{22})}}$ a_1a_2 is positive and d is negative, so t_1t_2 is negative or zero. Hence at any point of this arc if t_1 changes sign, then so does t_2 (and vice versa) - say there are m_1 such simultaneous sign changes. Similarly, there are m_3 simultaneous sign changes of t_1 and t_2 on the arc $\overline{\overline{(\xi^{13};\xi^{21})}}$. On $\overline{\overline{(\xi^{22};\xi^{13})}}$ both a_1a_2 and d are positive, so t_1t_2 is positive or zero. Thus if t_1 changes sign, then so does t_2 – say there are m_2 such sign changes.

On $\overline{\overline{(\xi^{11};\xi^{21})}}$ t_1 and t_2 each change sign $m_1+m_2+m_3+1$ times. The signs of t_1 and t_2 at ξ^{11} are the same as at ξ^{21}, so $m_1+m_2+m_3$ is odd. On all the other arcs at least one of a_1 and a_2 is positive, so at least one of t_1 and t_2 is nonnegative - thus the simultaneous sign changes of t_1 and t_2 occur only at the indicated $m_1+m_2+m_3$ points.

Now let

$$t_1 = u_1q_1\dots q_kr_1\dots r_l \text{ and } t_2 = u_2q_1\dots q_kr'_1\dots r'_m$$

be factorizations of t_1 and t_2 into irreducibles, where $u_1,u_2 \in \mathbb{Q}^*$ and $q_1,\dots,q_k$ are the only prime factors of both t_1 and t_2. The simultaneous sign changes occur at the points of intersection of q_i with $\mathcal{C}$. Since, for each q_i, there is an even number of such points, and, for $i \neq j$, q_i and q_j intersect $\mathcal{C}$ in different points, $m_1+m_2+m_3$ is even – a contradiction.

It remains to show that $P(\underline{a})$ is valid for $(X_{v_{p_i}}, G_{v_{p_i}})$, $i \in \{1,\dots,6\}$. Take the substitutions

	p_1	p_2	p_3	p_4	p_5	p_6
t_1	1	d	1	1	1	1
t_2	d	1	d	1	1	1

Consider, for example, the case of the subspace $X_{v_{p_1}} = \{Q^+_{\xi^{11}}, Q^-_{\xi^{11}}, Q^+_{\xi^{21}}, Q^-_{\xi^{21}}\}$. If $t_1 = 1$ and $t_2 = d$ then obviously $t_1 \in D(1,a_1)$ and $dt_1t_2 \in D(1,a_1a_2)$, and it suffices to show that

$d = t_2 \in D(1, a_2)$ in the subspace $X_{v_{p_1}}$, that is, for $j \in \{1, 2\}$, and for $\mu = +$ or $\mu = -$

$$d \in Q^{\mu}_{\xi^{j1}} \text{ or } da_2 \in Q^{\mu}_{\xi^{j1}}.$$

Since $da_2 = -p_2p_3p_4p_5$, and since da_2 is nonnegative at points ζ of $\mathcal{C}$ sufficiently close to ξ^{j1}, it follows that da_2 is an element of both $Q^{+}_{\xi^{j1}}$ and $Q^{-}_{\xi^{j1}}$, for $j \in \{1, 2\}$.

We proceed in a similar manner with the remaining substitutions. Note that, by Theorem 3.3.4, $P(\underline{a})$ holds true for every proper subspace of (X_F, G_F). $\square$

3.4. The pp conjecture for function fields of two parallel lines over $\mathbb{Q}$

In this section we complete our analysis by considering the case of a real irreducible two parallel lines, that is $ax^2 + c = 0$, $a > 0$, $c < 0$. We might as well assume $a = 1$. This case is similar to the elliptic case, and the main arguments and results from the previous section carry over, with a bit of modification here and there.

The coordinate ring $\mathbb{Q}[\mathcal{C}]$ can be identified with $\mathbb{Q}(\sqrt{-c})[y]$, the polynomial ring in one variable y with coefficients in the field $\mathbb{Q}(\sqrt{-c})$. The valuations that are of interest to us are also easy to describe. Units are identified with non-zero elements of $\mathbb{Q}(\sqrt{-c})$. Unlike what happens in the elliptic case, units no longer necessarily have constant sign on $\mathcal{C}$.

We still have the linear irreducibles $p = rx + sy + t$, $r, s, t \in \mathbb{Q}$, $s \neq 0$, but these no longer suffice. To copy certain of the constructions used in the proofs of Theorems 3.3.4 and 3.3.5, we also use the fact that there are enough quadratic irreducibles in $\mathbb{Q}[\mathcal{C}]$ of the form

$$p = x \pm (r(y + s)^2 + t),\ r, s, t \in \mathbb{Q},\ r > 0,\ |t| < \sqrt{-c}.$$

3.4.1. Lemma. *(*[**GłaMar-1**, Lemma 7]*) For given real r, s, t satisfying $r > 0$, $|t| < \sqrt{-c}$, there exist rationals r', s' and t' arbitrarily close to r, s and t respectively, such that $x + (r'(y + s')^2 + t')$ and $x - (r'(y + s')^2 + t')$ are irreducible in $\mathbb{Q}[\mathcal{C}]$.*

Proof. The discriminant of $\sqrt{-c} \pm (r'(y + s')^2 + t') \in \mathbb{Q}(\sqrt{-c})[y]$ is $-4r'(t' \pm \sqrt{-c})$. We want this to be not a square in $\mathbb{Q}(\sqrt{-c})$. Proceed as follows: choose r' to be any rational square close to r, choose s' close to s, choose t' close to t and such that ${t'}^2 + c$ is not a rational square (so then $-t' - \sqrt{-c}$ and $-t' + \sqrt{-c}$ are not squares in $\mathbb{Q}(\sqrt{-c})$). We can, for example, choose t' of the form $t' = q^k t_1$ where q is a prime such that the value of $-c$ at q is odd, $2k > v_q(-c)$ and $v_q(t_1) \geq 0$. Then $v_q(t'^2 + c) = v_q(-c)$ is odd, so $t'^2 + c$ is not a square. $\square$

The correspondence between points on $\mathcal{C}$ and orderings on $\mathbb{Q}(\mathcal{C})$ is the same as before, but now there are additional orderings corresponding to the four half-branches of $\mathcal{C}$ at ∞, namely the orderings compatible with the real valuation v_∞ on $\mathbb{Q}(\mathcal{C})$ defined by $v_\infty(f) = -\deg_y(f)$.

Lemma 3.3.3 carries over with the same proof. Using this, we see that an irreducible p has an even (odd, respectively) number of roots on the line $x = -\sqrt{-c}$, and also on the line $x = \sqrt{-c}$, if $\deg_y(p)$ is even (if $\deg_y(p)$ is odd, respectively).

When applying Theorem 3.1.4, we shall note the following: suppose f, g, h are non-zero elements of $\mathbb{Q}[\mathcal{C}]$. Then $f \in D(g,h)$ holds in $(X_{v_\infty}, G_{v_\infty})$ if and only if $fg \geq 0$ at ζ or $fh \geq 0$ at ζ holds for all real points $\zeta = (\pm\sqrt{-c}, \zeta_2)$ of $\mathcal{C}$ with $|\zeta_2|$ sufficiently large.

With these preliminary remarks out of the way, we now state the main results:

3.4.2. Theorem. *([**GłaMar-1**, Theorem 8]) Let $\mathcal{C}$ be the irreducible rational curve $x^2+c = 0$ without rational points and with $c < 0$, let F denote its function field. For a given pp-formula $P(\underline{a})$, where $\underline{a} = (a_1, \dots, a_k)$, let Σ denote the set of all irreducible factors of $a_1, \dots, a_k$. The following conditions are equivalent:*

(1) *$P(\underline{a})$ holds true in (X_{v_p}, G_{v_p}), for every $p \in \Sigma \cup \{\infty\}$,*
(2) *$P(\underline{a})$ holds true in every proper subspace of (X_F, G_F),*
(3) *$P(\underline{a})$ holds true in every finite subspace of (X_F, G_F).*

The proof of Theorem 3.4.2 is the same as the proof of Theorem 3.3.4, with minor modifications to allow for the fact that we are now dealing with two parallel lines. In defining the p_{I_j} we allow not only linear irreducibles, but also quadratic irreducibles (to take care of the case where the intervals I_j and J are both on the same component of $\mathcal{C}$). In the definition of the t'_{i1}, we define $t'_{i1} = \mu_i \prod_{j=1}^{m} {p_{I_j}}^{\theta_{ij}}$, where $\theta_{ij} = 0$ or 1 depending on whether t_i has the same sign or opposite sign at the opposite ends of the open interval I_j, and where $\mu_i \in \{1, -1, x, -x\}$ is chosen so that t'_{i1} has the same sign as t_i at the ends of each of the intervals I_j.

3.4.3. THEOREM. *([**GłaMar-1**, Theorem 9]) Let $\mathcal{C}$ be the irreducible rational curve $x^2+c=0$ without rational points and with $c<0$, let F be its function field. There exists a pp formula $P(\underline{a})$ which holds for every proper subspace of (X_F, G_F), but fails in (X_F, G_F).*

Again, the proof of Theorem 3.4.3 is analogous to the proof of Theorem 3.3.5, but instead of using just linear irreducibles we also allow suitably chosen quadratic irreducibles. We arrange the zeros ξ^{1i}, ξ^{2i}, $i \in \{1, \ldots, 6\}$ of these six irreducibles as in Fig. 2.

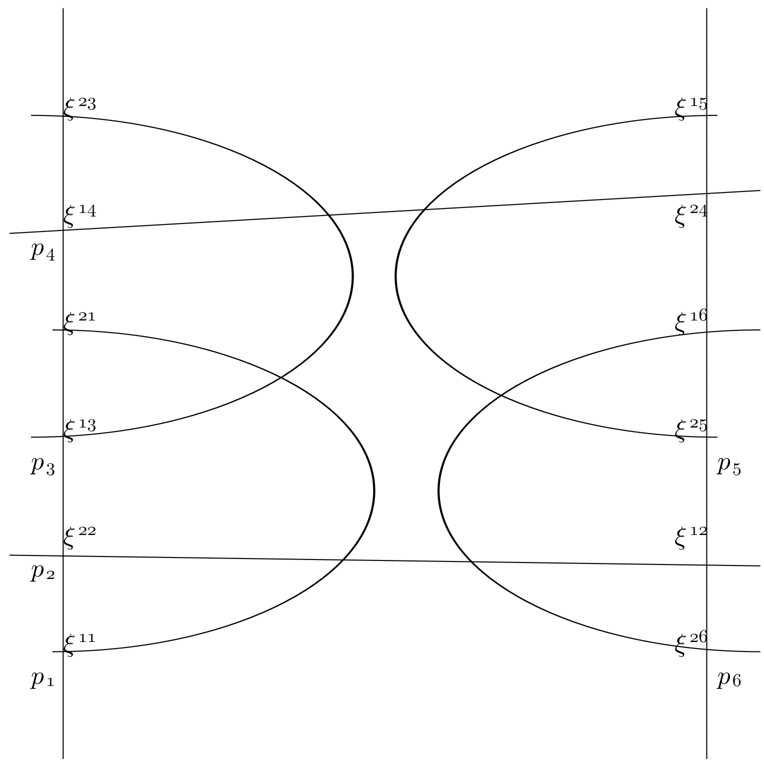


Fig. 2

3.5. The pp conjecture for the field $\mathbb{Q}(x_1, \ldots, x_n)$

As an application of the results obtained above, we shall prove the following result:

3.5.1. THEOREM. *The pp conjecture fails for some pp formula in the space of orderings of $\mathbb{Q}(x_1, \ldots, x_n)$, $n \geq 2$.*

PROOF. We proceed by induction. If $n=2$, then let $f(x_1, x_2) = 0$ be an equation of an irreducible conic section without rational points, for example let $f(x_1, x_2) = x_1^2 + x_2^2 - 3$. Then the space (X_{v_f}, G_{v_f}) of orderings compatible with the valuation v induced by f is a subspace of the space $(X_{\mathbb{Q}(x_1,x_2)}, G_{\mathbb{Q}(x_1,x_2)})$. Moreover, this space is also a group extension of the space of orderings of the residue field $\mathbb{Q}(x_1,x_2)_{v_f}$, that is the function field of the curve $f(x_1,x_2)=0$. If the pp conjecture was true for every pp formula in the space $(X_{\mathbb{Q}(x_1,x_2)}, G_{\mathbb{Q}(x_1,x_2)})$, then it would also the case in the space (X_{v_f}, G_{v_f}) (Theorem 2.2.1) and, consequently, in the space $(X_{\mathbb{Q}(x_1,x_2)_v}, G_{\mathbb{Q}(x_1,x_2)_v})$ (Lemma 2.2.6), which is a contradiction.

For $n \geq 3$ consider the valuation $v : \mathbb{Q}(x_1,\ldots,x_n) \to \mathbb{Z} \cup \{\infty\}$ given by $v(f) = p$ for $0 \neq f \in \mathbb{Q}(x_1,\ldots,x_n)$ such that $f = x_n^p \cdot g$ and $x_n \nmid g$, or $v(f) = \infty$ for $f = 0$. Then the residue field of v is equal to $\mathbb{Q}(x_1,\ldots,x_{n-1})$, the pp conjecture fails in its space of orderings for some pp formula, and hence fails in the subspace of the space $(X_{\mathbb{Q}(x_1,\ldots,x_n)}, G_{\mathbb{Q}(x_1,\ldots,x_n)})$ containing orderings compatible to v, and, in turn, fails in $(X_{\mathbb{Q}(x_1,\ldots,x_n)}, G_{\mathbb{Q}(x_1,\ldots,x_n)})$. $\square$

CHAPTER 4

The space of orderings of the field $\mathbb{R}(x, y)$

The content of this chapter is essentially the same as the material presented in our work [**GłaMar-2**]: we show that there exists a pp formula for which the pp conjecture is not valid in the space of orderings of the field $\mathbb{R}(x, y)$. Contrary to Chapter 3, we do not use valuation theory here, but we refer to results stated in Chapter 2. We conclude this chapter with some general remarks and point out possible further directions of the research.

4.1. The pp conjecture for the field $\mathbb{R}(x, y)$

We shall prove the following result:

4.1.1. Theorem. *(*[**GłaMar-2**, Theorem 1]*) The pp conjecture fails for some pp formula in the space of orderings $(X_{\mathbb{R}(x,y)}, G_{\mathbb{R}(x,y)})$.*

Proof. For $\epsilon > 0$ consider the subspaces

$$X_\epsilon = U(x^2 + y^2 - 1) \cap U(1 + \epsilon - x^2 - y^2)$$

and let $G_\epsilon = G_{\mathbb{R}(x,y)}|_{X_\epsilon}$. Define the subspace

$$X = \bigcap_{\epsilon > 0} X_\epsilon$$

and let $G = G_{\mathbb{R}(x,y)}|_X$. By Theorem 2.2.1, it is sufficient to show that the conjecture fails in the space (X, G). For $\epsilon > 0$ denote

$$A_\epsilon = \{(a, b) \in \mathbb{R}^2 : 1 < a^2 + b^2 < 1 + \epsilon\}$$

and let $\pi_1, \ldots, \pi_6 \in \mathbb{R}(x, y)$ be linear irreducibles which, for ϵ small enough, intersect with rings A_ϵ as in the following diagram:

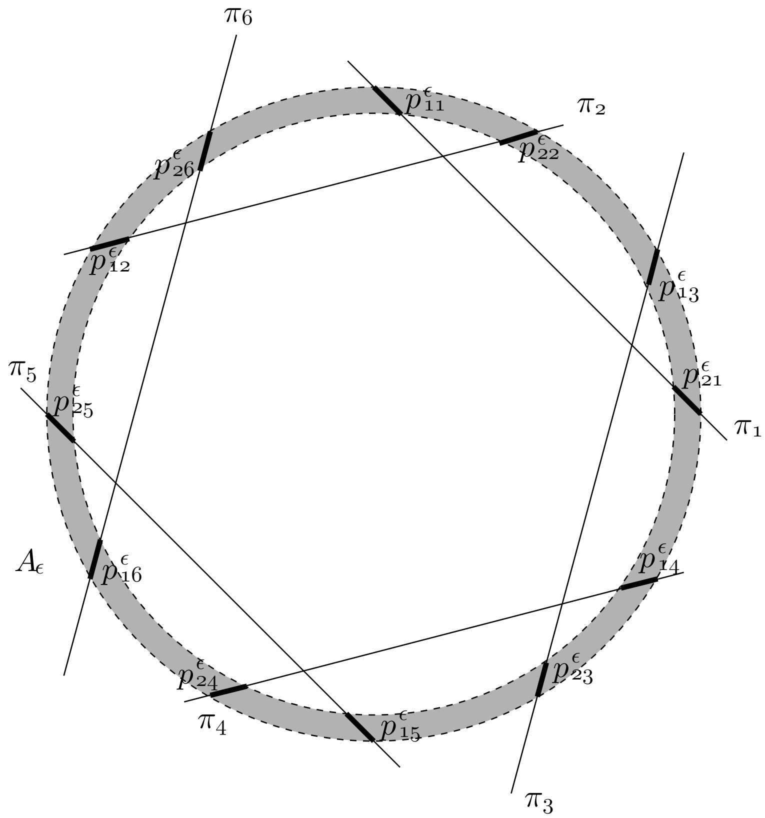


Fig. 3

Here $p_{1i}^{\epsilon}, p_{2i}^{\epsilon}$ denote the two connected components of $\mathcal{Z}(\pi_i) \cap A_{\epsilon}$, $i \in \{1, \ldots, 6\}$, $\epsilon > 0$, and are arranged in the above order, where $\mathcal{Z}(\pi_i)$ is the set of real zeros of π_i. Replacing π_i by $-\pi_i$ we may assume that every π_i is positive at the origin. For two adjacent line segments $p_{i_1 j_1}^{\epsilon}$ and $p_{i_2 j_2}^{\epsilon}$, $i_1, i_2 \in \{1, 2\}$, $j_1, j_2 \in \{1, \ldots, 6\}$, denote also by $A_n^{i_1 j_1, i_2 j_2}$ the ring sector starting at $p_{i_1 j_1}^{\epsilon}$ and, when moving clockwise along A_{ϵ}, ending at $p_{i_2 j_2}^{\epsilon}$.

Let $a_1 = \pi_1 \pi_6$, $a_2 = \pi_1 \pi_4$ and $d = -\pi_1 \pi_2 \pi_3 \pi_5$. Consider the following pp formula:

$$P(a_1, a_2, d) = \exists t_1 \exists t_2 (t_1 \in D(1, a_1) \wedge t_2 \in D(1, a_2) \wedge d t_1 t_2 \in D(1, a_1 a_2)).$$

We shall show that $P(a_1, a_2, d)$ fails to hold in the space (X, G).

Suppose, *a contrario*, that the formula holds true in (X, G) with certain $t_1, t_2 \in G$ verifying it. Without loss of generality we may assume that t_1, t_2 are square-free polynomials. Let

$$S = \{\sigma : \sigma \text{ is irreducible and } \sigma | t_1 \text{ or } t_2, \text{ or } \sigma = \pi_i \text{ for some } i \in \{1, \ldots, 6\}\}.$$

Observe, that there exists $\epsilon_1 > 0$ such that for $\epsilon_1 > \epsilon > 0$:

> for each $\sigma \in S$ the set $\mathcal{Z}(\sigma) \cap A_\epsilon$ is a finite disjoint union of smooth arcs $\gamma : (0,1) \to \mathbb{R}^2$ homeomorphic to an open line segment and such that $\lim_{t\to 0} \gamma(t)$ is a point on the circle $x^2+y^2 = 1$, whilst $\lim_{t\to 1} \gamma(t)$ is a point on $x^2+y^2 = 1+\epsilon$,

and

> for $\sigma, \tau \in S$, $\sigma \neq \tau$:[1]
> $$\mathcal{Z}(\sigma) \cap \mathcal{Z}(\tau) \cap A_\epsilon = \emptyset.$$

This is intuitively clear, however if one wants to prove it formally, one should use the "half-branches" theorem [**BCR**, Proposition 9.5.1], the fact that we may restrict ourselves to those $\sigma \in S$ for which ideals (σ) are real (see [**BCR**, Theorem 4.5.1]), and the fact that the distance from a semialgebraic set, as a continuous and semialgebraic function, is bounded on a closed and bounded set, and reaches its bounds ([**BCR**, Theorem 2.5.8]).

Observe also that, for $\epsilon > 0$ sufficiently small (say, $\epsilon_2 > \epsilon > 0$ for some $\epsilon_2 > 0$), $P(a_1, a_2, d)$ already holds in the subspace (X_ϵ, G_ϵ). Indeed, consider the open set

$$U = (U(-a_1) \cup U(t_1)) \cap (U(-a_2) \cup U(t_2)) \cap (U(-a_1a_2) \cup U(dt_1t_2)),$$

viewed as a subset in $X_{\mathbb{R}(x,y)}$. Since

$$t_1 \in D(1, a_1) \wedge t_2 \in D(1, a_2) \wedge dt_1t_2 \in D(1, a_1a_2)$$

holds true in (X, G), $X \subset U$. But $X = \bigcap_{\epsilon>0} X_\epsilon$ is a nested intersection of closed sets X_ϵ, and $X_{\mathbb{R}(x,y)}$ is compact, so for $\epsilon > 0$ small enough $X_\epsilon \subset U$. That means that $P(a_1, a_2, d)$ holds true in (X_ϵ, G_ϵ).

Fix $\epsilon > 0$ satisfying all of the above conditions (that is $\epsilon \leq \min\{\epsilon_1, \epsilon_2\}$) and consider the space (X_ϵ, G_ϵ). By looking at number of sign changes of each irreducible factor σ of t_1 or t_2 when we travel around the annulus region A_ϵ, we see that each such $\mathcal{Z}(\sigma)$ intersects with A_ϵ in an even number of connected components [**BCR**, Theorem 4.5.1].

[1] Note that some of $\pi_1, \ldots, \pi_6$ might be also divisors of t_1 or t_2

Furthermore, the signs of a_1, a_2 and d on the ring sectors between the successive p_{ij}^{ϵ}, $i \in \{1, 2\}$, $j \in \{1, \ldots, 6\}$, are the following:

	$A_{\epsilon}^{11,22}$	$A_{\epsilon}^{22,13}$	$A_{\epsilon}^{13,21}$	$A_{\epsilon}^{21,14}$	$A_{\epsilon}^{14,23}$	$A_{\epsilon}^{23,15}$	$A_{\epsilon}^{15,24}$	$A_{\epsilon}^{24,16}$	$A_{\epsilon}^{16,25}$	$A_{\epsilon}^{25,12}$	$A_{\epsilon}^{12,26}$	$A_{\epsilon}^{26,11}$
a_1	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$
a_2	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$
d	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$

We obtain a contradiction by investigating the behavior of t_1 and t_2 on A_{ϵ}. Note that Theorem 3.1.4 in this setup reads as follows:

$$f \in D_{X_{\epsilon}}(1, g) \Leftrightarrow \forall (a, b) \in A_{\epsilon}[f(a, b) \geq 0 \text{ or } f(a, b) \cdot g(a, b) \geq 0].$$

On $A_{\epsilon}^{21,14}$, $A_{\epsilon}^{24,16}$ and $A_{\epsilon}^{26,11}$ both a_1 and a_2 are positive, so t_1 and t_2 are nonnegative. Moreover, since t_1 and t_2 are square-free and since there are no singular points of irreducible factors of t_1, t_2 inside of A_{ϵ}, by the Sign Changing Criterion [**BCR**, Theorem 4.5.1], t_1 and t_2 are, in fact, positive.

Near p_{23}^{ϵ} a_1 is positive, so t_1 is positive. It follows that $\mathcal{Z}(t_1)$ (from now on we shall simply write t_1) does not intersect with A_{ϵ} along p_{13}^{ϵ}: if it did, then π_3 would divide t_1 (since they would have infinitely many points in common), so $t_1 = 0$ on p_{23}^{ϵ}.

Furthermore, $a_1 a_2 > 0$ near p_{13}^{ϵ}, so $d t_1 t_2$ is nonnegative. Since d changes sign between $A_{\epsilon}^{22,13}$ and $A_{\epsilon}^{13,21}$, and t_1 does not intersect with A_{ϵ} along p_{13}^{ϵ}, t_2 has to pass A_{ϵ} at p_{13}^{ϵ}. Thus $\pi_3 | t_2$ and t_2 also cuts across A_{ϵ} at p_{23}^{ϵ}.

Similarly, $a_2 > 0$ near p_{12}^{ϵ}, so $t_2 > 0$ and, as before, t_2 does not intersect with A_{ϵ} along p_{22}^{ϵ}. Close to p_{22}^{ϵ}, $a_1 a_2 > 0$, so $d t_1 t_2 \geq 0$ and thus t_1 passes A_{ϵ} at p_{22}^{ϵ} and also at p_{12}^{ϵ}.

Next, near p_{11}^{ϵ} $a_1 a_2 > 0$, so $d t_1 t_2 \geq 0$, whilst d changes sign between $A_{\epsilon}^{26,11}$ and $A_{\epsilon}^{11,22}$. Thus $t_1 t_2$ changes sign, so either t_1 intersects with A_{ϵ} along p_{11}^{ϵ} and t_2 does not, or t_2 does and t_1 does not.

Similarly, near p_{21}^{ϵ} $a_1 a_2 > 0$, so $d t_1 t_2 \geq 0$. d changes sign at p_{21}^{ϵ} and so does $t_1 t_2$, which implies that either t_1 crosses A_{ϵ} at p_{21}^{ϵ} and t_2 does not, or t_1 does not cross and t_2 does.

Of course if t_1 passes A_{ϵ} at p_{11}^{ϵ}, then $\pi_1 | t_1$, so t_1 also passes A_{ϵ} at p_{21}^{ϵ}. Therefore t_1 cuts across A_ϵ at p^ϵ_{11} if and only if it cuts across A_ϵ at p^ϵ_{21} and, similarly, t_2 traverses A_ϵ at p^ϵ_{11} if and only if it traverses A_ϵ at p^ϵ_{21}.

On $A^{11,22}_\epsilon$ $a_1a_2 > 0$, so $dt_1t_2 \geq 0$. Since $d < 0$, $t_1t_2 \leq 0$, so t_1 intersects with A_ϵ if and only if t_2 does – say, there are m_1 such intersections within $A^{11,22}_\epsilon$.

Similarly, on $A^{13,21}_\epsilon$ $a_1a_2 > 0$, so $dt_1t_2 \geq 0$. At the same time $d < 0$, so $t_1t_2 \leq 0$. Thus t_1 intersects with A_ϵ if and only if t_2 does; there are m_2 such intersections within $A^{13,21}_\epsilon$.

Finally, on $A^{22,13}_\epsilon$ $a_1a_2 > 0$ and $d > 0$, so $dt_1t_2 \geq 0$ and $t_1t_2 \geq 0$. Therefore t_1 intersects with A_ϵ if and only if t_2 does and we have m_3 such simultaneous intersections within $A^{22,13}_\epsilon$.

To sum up, there are $m_1 + m_2 + m_3$ simultaneous intersections of t_1 and t_2 with A_ϵ in $A^{11,21}_\epsilon$. Furthermore, t_1 crosses through p^ϵ_{22} and t_2 through p^ϵ_{13}. And finally, exactly one of t_1, t_2 crosses through both p^ϵ_{11} and p^ϵ_{21}: say t_i does and t_j does not. Then t_j changes sign $m_1 + m_2 + m_3 + 1$ times from $A^{26,11}_\epsilon$ to $A^{21,14}_\epsilon$, to go from positive to positive, hence $m_1 + m_2 + m_3 + 1$ is even and $m_1 + m_2 + m_3$ is odd.

Note now that the only simultaneous intersections of t_1 and t_2 with A_ϵ are the $m_1+m_2+m_3$ listed above; on all other sectors of A_ϵ at least one of a_1, a_2 is positive, forcing either t_1 or t_2 to be positive as well.

Simultaneous intersections may occur only at the common irreducible factors of t_1, t_2, and two distinct irreducibles cannot intersect in the same place. According to our assumptions, each such factor has an even number of crossings with A_n – so $m_1 + m_2 + m_3$ is even, which is a contradiction. This finishes the first half of the proof.

It remains to show that $P(a_1, a_2, d)$ holds true on every finite subspace of (X, G). Suppose then that there is a finite subspace Y of (X, G) on which $P(a_1, a_2, d)$ fails to hold, and denote for simplicity $H = G|_Y$. Without loss of generality we may assume that (Y, H) is minimal with such property. We need to consider two cases.

Firstly, suppose that $d \notin D((1, a_1) \otimes (1, a_2))$ holds on (Y, H). We shall use the description of value sets of Pfister forms stated in Lemma 2.2.3. Thus, for some $\sigma \in Y$, $a_1\sigma = 1$, $a_2\sigma = 1$ and $d\sigma = -1$. Clearly $\sigma \in X_\epsilon$ for any fixed $\epsilon > 0$, so, by the Tarski Transfer Principle [**BCR**, Corollary 5.2.4], there is a point $(a, b) \in A_\epsilon$ such that $a_1(a, b) > 0$, $a_2(a, b) > 0$ and $d(a, b) < 0$. But there is no such point in A_ϵ (see the table) – a contradiction.

Now assume that $d \in D((1, a_1) \otimes (1, a_2))$ holds in (Y, H). By Lemma 2.4.4 there exists a connected component (Y_0, H_0) of (Y, H), which is not a fan, such that, if $(\overline{Y}, \overline{H})$ denotes the residue space of (Y_0, H_0), $a_1, a_2 \in \overline{H}$, neither a_1, a_2 nor $a_1 a_2$ is equal to -1, $(1, a_1) \otimes (1, a_2)$ is isotropic over (Y_0, H_0) and $d \notin \overline{H}$. Clearly $P(a_1, a_2, d)$ already fails to hold in (Y_0, H_0), so, due to minimality of (Y, H), $(Y, H) = (Y_0, H_0)$.

Since $a_1, a_2, a_1 a_2 \neq -1$, there are elements of $\overline{Y}$ making a_1, a_2 and $a_1 a_2$ positive. At the same time, since $(1, a_1) \otimes (1, a_2)$ is isotropic, there is no element of $\overline{Y}$ making both a_1 and a_2 positive. There exist $\sigma_1, \sigma_2, \sigma_3 \in \overline{Y}$ such that a_1, a_2 and $a_1 a_2$ have the following signs:

	σ_1	σ_2	σ_3
a_1	+	-	-
a_2	-	+	-
$a_1 a_2$	-	-	+

Consider the subspace $(\tilde{Y}, \tilde{H})$ for which $\{\sigma_1, \sigma_2, \sigma_3\}$ is a minimal generating set. $(\tilde{Y}, \tilde{H})$ is not a fan, since $\overline{Y}$ has no orderings making a_1 and a_2 both positive. Thus elements of $\tilde{Y}$, viewed as characters, are products $\prod_{i=1}^{3} \sigma_i^{e_i}$ such that $\sum_{i=1}^{3} e_i \equiv 1 \mod 2$ and do not contain the element $\sigma_1 \sigma_2 \sigma_3$ and, consequently, $\tilde{Y} = \{\sigma_1, \sigma_2, \sigma_3\}$. Let (Y_1, H_1) be the group extension of $(\tilde{Y}, \tilde{H})$ where $H_1 = \tilde{H}[d]$. It consists of 6 orderings σ_1^+, σ_2^+, σ_3^+, σ_1^-, σ_2^-, σ_3^-, with respect to which the signs of $a_1, a_2, a_1 a_2, d$ are as follows:

	σ_1^+	σ_2^+	σ_3^+	σ_1^-	σ_2^-	σ_3^-
a_1	+	-	-	+	-	-
a_2	-	+	-	-	+	-
$a_1 a_2$	-	-	+	-	-	+
d	+	+	+	-	-	-

$P(a_1, a_2, d)$ fails to hold on (Y_1, H_1), so $(Y, H) = (Y_1, H_1)$.

Define the following subspaces of (X, G):

$$
\begin{aligned}
V^{11,22} &= U(-\pi_1) \cap U(-\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{22,13} &= U(-\pi_1) \cap U(\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{13,21} &= U(-\pi_1) \cap U(\pi_2) \cap U(-\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{21,14} &= U(\pi_1) \cap U(\pi_2) \cap U(-\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(\pi_6)
\end{aligned}
$$

$$
\begin{aligned}
V^{14,23} &= U(\pi_1)\cap U(\pi_2)\cap U(-\pi_3)\cap U(-\pi_4)\cap U(\pi_5)\cap U(\pi_6)\\
V^{23,15} &= U(\pi_1)\cap U(\pi_2)\cap U(\pi_3)\cap U(-\pi_4)\cap U(\pi_5)\cap U(\pi_6)\\
V^{15,24} &= U(\pi_1)\cap U(\pi_2)\cap U(\pi_3)\cap U(-\pi_4)\cap U(-\pi_5)\cap U(\pi_6)\\
V^{24,16} &= U(\pi_1)\cap U(\pi_2)\cap U(\pi_3)\cap U(\pi_4)\cap U(-\pi_5)\cap U(\pi_6)\\
V^{16,25} &= U(\pi_1)\cap U(\pi_2)\cap U(\pi_3)\cap U(\pi_4)\cap U(-\pi_5)\cap U(-\pi_6)\\
V^{25,12} &= U(\pi_1)\cap U(\pi_2)\cap U(\pi_3)\cap U(\pi_4)\cap U(\pi_5)\cap U(-\pi_6)\\
V^{12,26} &= U(\pi_1)\cap U(-\pi_2)\cap U(\pi_3)\cap U(\pi_4)\cap U(\pi_5)\cap U(-\pi_6)\\
V^{26,11} &= U(\pi_1)\cap U(-\pi_2)\cap U(\pi_3)\cap U(\pi_4)\cap U(\pi_5)\cap U(\pi_6).
\end{aligned}
$$

By the Tarski Transfer Principle subspaces $V^{i_1j_1,i_2j_2}$ form a partition of (X,G) and, clearly, signs of a_1, a_2 and d on the $V^{i_1j_1,i_2j_2}$ are exactly the same as on the sector $A_\epsilon^{i_1j_1,i_2j_2}$, for respective i_1, i_2, j_1, j_2. Comparing those signs we see that $\sigma_1^- \in V^{23,15}$, and $(\sigma_1^+ \in V^{14,23} \vee \sigma_1^+ \in V^{15,24})$, and $\sigma_2^- \in V^{25,12}$, and $(\sigma_2^+ \in V^{16,25} \vee \sigma_2^+ \in V^{12,26})$, and $\sigma_3^+ \in V^{22,13}$, and, finally, $(\sigma_3^- \in V^{11,22} \vee \sigma_3^- \in V^{13,21})$.

Consider the following two 4-element fans:

$$\{\sigma_1^+,\sigma_1^-,\sigma_2^+,\sigma_2^-\} \text{ and } \{\sigma_1^+,\sigma_1^-,\sigma_3^+,\sigma_3^-\}.$$

If $\sigma_1^+ \in V^{14,23}$ and $\sigma_2^+ \in V^{12,26}$, then, in particular, $\pi_3(\sigma_1^+\sigma_1^-\sigma_2^+\sigma_2^-) = -1$, which is a contradiction, since for every 4-element fan $\{\rho_1,\dots,\rho_4\}$ $\prod_{i=1}^4 \rho_i = 1$ (note that we can also use π_2 instead of π_3). On the other hand, if $\sigma_1^+ \in V^{14,23}$ and $\sigma_2^+ \in V^{16,25}$, then $\pi_5(\sigma_1^+\sigma_1^-\sigma_2^+\sigma_2^-) = -1$, which is again a contradiction. Thus $\sigma_1^+ \in V^{15,24}$.

If $\sigma_1^+ \in V^{15,24}$ and $\sigma_3^- \in V^{13,21}$, then $\pi_3(\sigma_1^+\sigma_1^-\sigma_3^+\sigma_3^-) = -1$: a contradiction. But if $\sigma_1^+ \in V^{15,24}$ and $\sigma_3^- \in V^{11,22}$, then $\pi_2(\sigma_1^+\sigma_1^-\sigma_3^+\sigma_3^-) = -1$, which eliminates the last case. □

4.2. Further remarks

Obviously spaces of orderings of function fields of rational conic sections and the space of orderings of the field $\mathbb{R}(x,y)$ do not exhaust all examples of spaces of orderings of stability index 2. By Lemma 1.7.3, spaces of orderings of formally real finitely generated algebraic extensions of $\mathbb{R}(x, y)$ are also of stability index 2 – we can think geometrically of those fields as of formally real function fields of algebraic surfaces in $\mathbb{R}^3$. One would expect the pp conjecture to fail for such spaces.

We can also ask a more general question: if F is a formally real finitely generated extension of $\mathbb{R}$ of transcendence degree at least two, is it true that that the pp conjecture fails in the space of orderings (X_F, G_F)? One would conjecture that it does, however so far we can only prove this in a special case. Namely, we have the following result:

4.2.1. THEOREM. *The pp conjecture fails for some pp formula in the space of orderings of* $\mathbb{R}(x_1, \ldots, x_n)$ *for* $n \geq 2$.

PROOF. We proceed by induction on n. For $n = 2$ this is precisely the main result presented in this chapter. For $n \geq 3$ we use the same method as in the proof of Theorem 3.5.1: consider the valuation $v : \mathbb{R}(x_1, \ldots, x_n) \to \mathbb{Z} \cup \{\infty\}$ given by $v(f) = p$ for $0 \neq f \in \mathbb{R}(x_1, \ldots, x_n)$ such that $f = x_n^p \cdot g$ and $x_n \nmid g$, or $v(f) = \infty$ for $f = 0$. Then the residue field of v is equal to $\mathbb{R}(x_1, \ldots, x_{n-1})$, by the inductive hypothesis the pp conjecture already fails for the space of orderings of $\mathbb{R}(x_1, \ldots, x_{n-1})$, and hence fails for the subspace of the space $(X_{\mathbb{R}(x_1,\ldots,x_n)}, G_{\mathbb{R}(x_1,\ldots,x_n)})$ containing orderings compatible to v, which is a group extension of $(X_{\mathbb{R}(x_1,\ldots,x_{n-1})}, G_{\mathbb{R}(x_1,\ldots,x_{n-1})})$. As a result, the pp conjecture fails for the space $(X_{\mathbb{R}(x_1,\ldots,x_n)}, G_{\mathbb{R}(x_1,\ldots,x_n)})$ by Theorem 2.2.1. $\square$

We remark that both Theorem 4.1.1 and Theorem 4.2.1 are still valid with the field $\mathbb{R}$ replaced by an arbitrary real closed field R; all results used to prove these theorems work equally well for any real closed field and for $\mathbb{R}$.

The question whether the pp conjecture holds for the space of orderings of the field of power series in two variables $\mathbb{R}((x, y))$ is also of considerable interest. If the conjecture failed for this space, one would be able to disprove it for spaces of orderings of power series in n variables over the reals, $n \geq 2$. This, in turn, might be a way of disproving the pp conjecture for spaces of orderings of formally real finitely generated extensions of $\mathbb{R}$ of transcendence degree n mentioned above, since the completion of the coordinate ring at a nonsingular point of a real algebraic variety in $\mathbb{R}^n$ is just the power series ring in n variables.

CHAPTER 5

Testing pp formulae on finite subspaces

In this chapter we prove Theorem 5.1.1, which was first stated in [**AstTre05**] – our proof uses the same methods, however it differs slightly in some details. For a pp formula $P(\underline{y})$ a special family of formulae $\mathcal{F}_P$ is defined and it is shown how this family can be used to test whether the pp formula fails on a finite subspace of every space of orderings. We then refine this result and give an explicit description of formulae in the family $\mathcal{F}_P$ that need to be tested in order to achieve the same result. In Section 2 we apply these results to the pp conjecture and, in particular, present a proof of Theorem 2.2.1 due to Astier and Tressl [**AstTre05**].

5.1. Families of testing formulae

Instead of investigating a pp formula in a fixed space of orderings, we shall rather consider it as an expression in the language L_{SG} of special groups. We work with a fixed pp formula

$$P(\underline{y}) = \exists \underline{t} \bigwedge_{j=1}^{m} \theta_j(\underline{t}, \underline{y}),$$

where θ_j are atomic formulae and $\underline{y} = (y_1, \ldots, y_k)$, $\underline{t} = (t_1, \ldots, t_n)$ are tuples of individual variables in the language L_{SG}. Define the family of spaces of orderings and constants:

$$\mathbb{K}_P = \{(Y, H, \underline{b}) : (Y, H) \text{ is a finite space of orderings, } \underline{b} \in H^k, P(\underline{b}) \text{ fails in } (Y, H),$$
$$P(\underline{b}) \text{ holds in every proper subspace of } (Y, H)\}$$

and the corresponding family of formulae in the language L_{SG}:

$$\mathcal{F}_P = \{Q(\underline{y}) = \forall \underline{s} \neg \bigwedge_{j=1}^{m'} \theta'_j(\underline{s}, \underline{y}) : \theta'_j \text{ are atomic formulae in the language } L_{SG}$$
$$\underline{s} = (s_1, \ldots, s_{n'}) \text{ is a tuple of individual variables, } n' \in \mathbb{N},$$
$$\forall (Y, H, \underline{b}) \in \mathbb{K}_P [Q(\underline{b}) \text{ holds in } (Y, H)]\}.$$

The family $\mathcal{F}_P$ consists of all formulae $Q(\underline{y})$ in the language L_{SG} having the form $Q(\underline{y}) = \neg R(\underline{y})$, where $R(\underline{y})$ is a pp formula such that $R(\underline{b})$ fails in (Y, H) for all $(Y, H, \underline{b}) \in \mathbb{K}_P$.

The following theorem is proven in [**AstTre05**]; we give a slightly different proof (in particular, the definition of $\mathbb{K}_P$ differs), although we essentially use the same methods.

5.1.1. Theorem. *Let (X, G) be a space of orderings, let $\underline{a} \in G^k$. The following two conditions are equivalent:*

(1) *$P(\underline{a})$ fails in some finite subspace of (X, G);*

(2) *for every $Q(\underline{y}) \in \mathcal{F}_P$ the formula $Q(\underline{a})$ holds in (X, G).*

Proof. (1) $\Rightarrow$ (2). By Zorn's Lemma there is a finite subspace (Y, H) of (X, G) such that $P(\underline{a}|_Y)$ fails in (Y, H) and holds on every proper subspace of (Y, H). Then $(Y, H, \underline{a}|_Y) \in \mathbb{K}_P$. Fix an $Q(\underline{y}) \in \mathcal{F}_P$ – we thus have that $Q(\underline{a}|_Y)$ holds in (Y, H). It follows that $Q(\underline{a})$ holds in (X, G), for if for some $\underline{s} \in G^{n'}$ all $\theta'_j(\underline{s}, \underline{a})$ would hold in (X, G), $j \in \{1, \ldots, m'\}$, then all $\theta'_j(\underline{s}|_Y, \underline{a}|_Y)$ would hold in (Y, H), $j \in \{1, \ldots, m'\}$, that is $Q(\underline{a}|_Y)$ would fail in (Y, H).

(2) $\Rightarrow$ (1). Let

$$\mathcal{T}(\mathbb{K}_P) = \{A(\underline{y}) : A(\underline{y}) \text{ is a formula in the free variables } \underline{y},$$
$$\forall (Y, H, \underline{b}) \in \mathbb{K}_P [A(\underline{b}) \text{ holds in } (Y, H)]\}.$$

In other words, $\mathcal{T}(\mathbb{K}_P)$ is the theory of $\mathbb{K}_P$ in the language L_{SG} extended by the constants $\underline{y}$.

We want to construct an SG-morphism π from the group G to a reduced special group H such that, for every $A(\underline{y}) \in \mathcal{T}(\mathbb{K}_P)$, $A(\underline{b})$, holds in (Y, H), (Y, H) denoting the space of orderings induced by H, and $\underline{b} = \pi(\underline{a})$. It suffices to construct a model H of $\{A(\underline{a}) : A(\underline{y}) \in \mathcal{T}(\mathbb{K}_P)\}$ (which includes axioms of the theory reduced special groups) in the language $\mathcal{L}$ of reduced special groups extended by the constants from G, in which all sentences

(14) $$g_1 = g_2 \cdot g_3, \qquad \text{for } g_1, g_2, g_3 \in G, \text{ and}$$

(15) $$g_1 \in D(g_2, g_3), \qquad \text{for } g_1, g_2, g_3 \in G,$$

which hold in (X, G), are also true.We will denote a model H of $\mathcal{T}(\mathbb{K}_P)$ by $(Y, H, \underline{b})$ to indicate the associated space of orderings (Y, H) and distinguished parameters $\underline{b}$. Note that every model of $\mathcal{T}(\mathbb{K}_P)$ is an element of $\mathbb{K}_P$ (by Lemma 2.2.2, there is a uniform bound on $|H|$ for $(Y, H, \underline{b}) \in \mathbb{K}_P$, and the conditions "$H$ has at most B elements", "$P(\underline{b})$ fails in (Y, H)" and "$P(\underline{b})$ holds in proper subspaces of (Y, H)" are expressible as formulae in $\mathcal{T}(\mathbb{K}_P)$).

Take any finite collection $S_1(\underline{g}_1), \ldots, S_s(\underline{g}_s)$ of atomic formulae (14) or (15). Some of the entries of the $\underline{g}_i$ may coincide with each other, or may be ± 1, or may coincide with entries of $\underline{a}$. Relabelling suitably, we can write each $S_i(\underline{g}_i)$ as $S_i(\underline{g}, \underline{a})$, where $\underline{g} = (g_1, \ldots, g_t)$, $g_1, \ldots, g_t \in G$ are distinct from each other, and from ± 1, and from the entries of $\underline{a}$. We also have the problem that some of the entries of $\underline{a}$ may be equal to each other or to ± 1. For each k, l such that $a_k = a_l$ we add the atomic formula "$a_k = a_l$" to our collection. Similarly, add "$a_k = 1$" (resp., "$a_k = -1$") if $a_k = 1$ (resp., $a_k = -1$). Define $Q(\underline{y})$ to be the pp formula $\exists \underline{u} \bigwedge_{i=1}^{s} S_i(\underline{u}, \underline{y})$. The formula $\bigwedge_{i=1}^{s} S_i(\underline{g}, \underline{a})$ holds in (X, G), which clearly implies that $Q(\underline{a})$ holds in (X, G). If $Q(\underline{b})$ fails for each $(Y, H, \underline{b})$ in the class $\mathbb{K}_P$, then $\neg Q(\underline{y})$ belongs to the class $\mathcal{F}_P$, so $\neg Q(\underline{a})$ holds in (X, G), by our assumptions. This is a contradiction. Thus $Q(\underline{b})$ holds for some $(Y, H, \underline{b}) \in \mathbb{K}_P$ with some $\underline{h} = (h_1, \ldots, h_t)$ verifying it. Fix such a $(Y, H, \underline{b})$. View (Y, H) as having constants from G by interpreting $g \in G$ to be the respective entry of $\underline{h}$, if g is an entry of $\underline{g}$, or to be b_k if $g = a_k$, or to be ± 1 if $g = \pm 1$, or to be some arbitrary element of H otherwise. By the compactness theorem, we have constructed the desired model.

Now the set $\{y \circ \pi : y \in Y\}$ is a generating set for a finite subspace of (X, G) and, since $P(\underline{b})$ fails in (Y, H), $P(\underline{a})$ fails in this subspace of (X, G). $\square$

We continue to work with the formula

$$P(\underline{y}) = \exists \underline{t} \bigwedge_{j=1}^{m} \theta_j(\underline{t}, \underline{y})$$

defined before. Let $\underline{x} = (x_1, \ldots, x_l)$ be a tuple of free variables in the language L_{SG}. We build a new formula $P_l(\underline{y}, \underline{x})$ by induction on l. If $l = 1$, we define $P_1(\underline{y}, x_1)$ by replacing each atomic formula $z_1 \in D(z_2, z_3)$ in $P(\underline{y})$ with

$$\exists s_1 \exists s_2 [(s_1 \in D(1, x_1)) \wedge (s_2 \in D(1, x_1)) \wedge (z_1 \in D(s_1 z_2, s_2 z_3))].$$

If $l \geq 2$, we define $P_l(\underline{y}, (x_1, \ldots, x_l))$ by performing the above action on $P_{l-1}(\underline{y}, (x_1, \ldots, x_{l-1}))$.

One sees that for each space of orderings (X, G) and for each subspace of the form $U(b_1, \ldots, b_l)$, $b_1, \ldots, b_l \in G$, if $\underline{a} \in G^k$ then $P_l(\underline{a}, \underline{b})$ holds in (X, G) if and only if $P(\underline{a})$ holds in the subspace $U(b_1, \ldots, b_l)$.

Let $\lambda \geq 1$ be an integer. We shall construct a sequence of formulae $P_\lambda^{(i)}(\underline{y})$, $i \geq 0$, by induction. For $i = 0$ we define $P_\lambda^{(0)}(\underline{y}) = P(\underline{y})$. For $i = 1$, we define

$$P_\lambda^{(1)} = \exists z_0 \ldots \exists z_\lambda \bigwedge_{j=1}^{\lambda} [(z_{j-1} \in D(1, z_j)) \wedge P_1(\underline{y}, z_{j-1}z_j)],$$

and for $i \geq 2$ we define $P_\lambda^{(i)}(\underline{y})$ by performing the above action on $P_\lambda^{(i-1)}(\underline{y})$ instead of $P(\underline{y})$. Note that this construction depends on λ.

Trivially, for every space of orderings (X, G) and every $\underline{a} \in G^k$, $P(\underline{a}) \Rightarrow P_\lambda^{(1)}(\underline{a})$ (by taking $z_0 = z_1 = \ldots = z_\lambda = 1$) and, consequently, $P(\underline{a}) \Rightarrow P_\lambda^{(1)}(\underline{a}) \Rightarrow \ldots \Rightarrow P_\lambda^{(i)}(\underline{a})$.

Define the number

$$c_P = \max\{\mathrm{cl}(X, G) : (X, G, \underline{a}) \in \mathbb{K}_P\};$$

by Lemma 2.2.2, this number is well defined. Moreover, the number c_P is uniformly bounded from the above by $B(n, k)$, although we do not claim that this bound is best possible. We shall prove the following result:

5.1.2. Theorem. *Let $\lambda > c_P$, let (X, G) be a space of orderings, let $\underline{a} \in G^k$. The following two conditions are equivalent:*

(1) *$P(\underline{a})$ fails in some finite subspace of (X, G);*
(2) *for every $i \geq 0$ the formula $\neg P_\lambda^{(i)}(\underline{a})$ holds in (X, G).*

Proof. (1) $\Rightarrow$ (2). By the "easy part" of the previous theorem, that is the implication (1) $\Rightarrow$ (2), it suffices to show that for every $i \geq 0$ $\neg P_\lambda^{(i)}(\underline{y}) \in \mathcal{F}_P$. Obviously $\neg P(\underline{y}) \in \mathcal{F}_P$, so $\neg P_\lambda^{(0)}(\underline{y}) \in \mathcal{F}_P$. Let $(Y, H, \underline{b}) \in \mathbb{K}_P$. Then $\mathrm{cl}(Y, H) \leq c_P < \lambda$, and hence, for every $c_0, \ldots, c_\lambda \in \mathbb{K}_P$ satisfying $c_{j-1} \in D(1, c_j)$, $j \in \{1, \ldots, \lambda\}$, there exists $j_0 \in \{1, \ldots, \lambda\}$ such that $c_{j_0-1}c_{j_0} = 1$. This forces $P_1(\underline{b}, c_{j_0-1}c_{j_0})$ to be logically equivalent to $P(\underline{b})$, which implies that the formula $\neg P^{(1)}_{\lambda}(\underline{b})$ holds in (Y,H) and, consequently, $\neg P^{(1)}_{\lambda}(\underline{a})$ holds in (X,G). From the construction of $P^{(i)}_{\lambda}(\underline{y})$, the argument follows for $i \geq 2$ by repeating the same reasoning.

(2) $\Rightarrow$ (1). Using Zorn's Lemma, choose a subspace (Y,H) of (X,G) minimal subject to the condition that for every $i \geq 0$ the formula $\neg P^{(i)}_{\lambda}(\underline{a})$ holds in (X,G). We shall show that $\mathrm{cl}(Y,H) < \lambda$. Suppose that, for some $c_0,\dots,c_\lambda \in G$, $c_{j-1} \in D(1,c_j)$ in (Y,H), and $c_{j-1}c_j \neq 1$ in (Y,H) for $j \in \{1,\dots,\lambda\}$. Define $Z_j = U(c_{j-1}c_j)\bigcap Y$; clearly $(Z_j, H|_{Z_j})$ are proper subspaces of (Y,H), so $\neg P^{(i)}_{\lambda}(\underline{a})$ fails in $(Z_j, H|_{Z_j})$ for $i \geq 0$, $j \in \{1,\dots,\lambda\}$, that is for every $j \in \{1,\dots,\lambda\}$ there is some $i \geq 0$ such that $P^{(i)}_{\lambda}(\underline{a})$ holds in $(Z_j, H|_{Z_j})$. Since $P^{(i)}_{\lambda}(\underline{a}) \Rightarrow P^{(i+1)}_{\lambda}(\underline{a})$ we may assume that for $i \geq 0$ big enough $P^{(i)}_{\lambda}(\underline{a})$ holds in $(Z_j, H|_{Z_j})$ for $j \in \{1,\dots,\lambda\}$. From the construction of $P^{(i+1)}_{\lambda}(\underline{y})$ it follows, that $P^{(i+1)}_{\lambda}(\underline{a})$ holds in (Y,H) – a contradiction. The result now follows, by Lemma 2.2.2. $\square$

5.2. Families of testing formulae and the pp conjecture

We continue to work with the formula $P(\underline{y})$ and families $\mathcal{F}_P$ and $\mathbb{K}_P$, as well as integers λ and c_P, defined as before. We shall investigate how the theorems proven in the previous section can be applied to the pp conjecture.

Firstly, Theorem 5.1.2 gives a concrete list of formulae that need to be verified in order to check if the pp conjecture holds true or not: for a space of orderings (X,G) and $\underline{a} \in G^k$ the formula $P(\underline{a})$ holds on each finite subspace of (X,G) if and only if the formula $P^{(i)}_{\lambda}(\underline{a})$ holds in (X,G) for some $i \geq 0$. One would like to have a better understanding of the formulae $P^{(i)}_{\lambda}(\underline{a})$ (or, in general, of the family $\mathcal{F}_P$) in the case when $P(\underline{y})$ is one of the examples of pp formulae discussed in previous chapters for which the pp conjecture fails. We note here that there is no claim that the subcollection $\{\neg P^{(i)}_{\lambda}(\underline{y}) : i \geq 0\}$ of $\mathcal{F}_P$ is all of $\mathcal{F}_P$.

In view of Theorem 5.1.2, for a space of orderings (X,G) and $\underline{a} \in G^k$, the assertion that if $P(\underline{a})$ holds in all finite subspaces of (X,G) then it also holds in (X,G), is equivalent to the statement that the formulae "$P^{(i)}_{\lambda}(\underline{a}) \Rightarrow P(\underline{a})$" hold in (X,G) for all $i \geq 0$. Namely, we have:

5.2.1. Theorem. *Let $\lambda > c_P$, let (X,G) be a space of orderings, let $\underline{a} \in G^k$. The following two conditions are equivalent:*

(1) *for each subspace (Y,H) of (X,G) the pp conjecture for $P(\underline{a})$ holds in (Y,H);*

(2) *for each subspace* (Y, H) *of* (X, G) *the formula "*$P^{(1)}_\lambda(\underline{a}) \Rightarrow P(\underline{a})$*" holds in* (Y, H).

Proof. The implication (1) $\Rightarrow$ (2) follows immediately from Theorem 5.1.2, so it remains to show that (2) $\Rightarrow$ (1). Suppose that the formula $P(\underline{a})$ fails on some subspace (Y, H) of (X, G), yet it holds true in every finite subspace of (Y, H), and assume that (Y, H) is minimal with this property. It follows that (Y, H) has infinite chain length, and thus there exist $c_0, \ldots, c_\lambda \in G$ such that $c_{j-1} \in D(1, c_j)$ on Y and $c_{j-1}c_j \neq 1$ on Y, $j \in \{1, \ldots, \lambda\}$. By the minimality of (Y, H), $P(\underline{a})$ holds true on $U(c_{j-1}c_j) \cap Y$, and, consequently, $P_1(\underline{a}, c_{j-1}c_j)$ holds on Y, $j \in \{1, \ldots, \lambda\}$. This implies that $P^{(1)}_\lambda(\underline{a})$ holds on Y, which contradicts (1). $\square$

Finally, we give the original Astier-Tressl proof of Theorem 2.2.1 – that is, we apply Theorem 5.1.1 to prove that the pp conjecture is preserved with respect to subspaces. Both here and in the previous proof of Theorem 2.2.1 Lemma 2.2.2 is used. It suffices to prove:

5.2.2. Lemma. *Let* (X, G) *be a space of orderings, let* $\underline{a} \in G^k$*, let* $Y = \bigcap_{b \in S} U(b)$ *be a subspace of* (X, G)*,* $S \subset G$*. If* $P(\underline{a})$ *holds true on every finite subspace of* Y*, then there exists a finite subset* $T \subset S$ *such that* $P(\underline{a})$ *holds true on every finite subspace of* $\bigcap_{b \in T} U(b)$.

Proof. By Theorem 5.1.1, there is a formula $Q(\underline{y}) \in \mathcal{F}_P$ such that $Q(\underline{a})$ fails on Y. Let

$$\neg Q(\underline{a}) = \exists \underline{s} \wedge_{j=1}^{m'} p'_j(\underline{s}, \underline{a}) \in D(1, q'_j(\underline{s}, \underline{a}))$$

where $p'_j(\underline{s}, \underline{a})$, $q'_j(\underline{s}, \underline{a})$ are $\pm$ products of some of the entries of $\underline{s}$ and $\underline{a}$, with $\underline{s} \in G^{n'}$ verifying $\neg Q(\underline{a})$. Consider the open set $U = \bigcap_{j=1}^{m'}(U(-q'_j(\underline{s}, \underline{a})) \cup U(p'_j(\underline{s}, \underline{a})))$. Since $\bigwedge_{j=1}^{m'} p'_j(\underline{s}, \underline{a}) \in D(1, q'_j(\underline{s}, \underline{a}))$ holds true in Y, $Y \subset U$. But $Y = \bigcap_{b \in S} U(b)$, so, by compactness, for some finite $T \subset S$, $\bigcap_{b \in T} U(b) \subset U$, which means that $Q(\underline{a})$ already fails in $\bigcap_{b \in T} U(b)$, and the conclusion follows from Theorem 5.1.1. $\square$

Observe that in view of the definition of the formulae $P_l(\underline{x}, \underline{y})$, the above lemma can be also stated in the following form:

5.2.3. Lemma. *Let* (X, G) *be a space of orderings, let* $\underline{a} \in G^k$*, let* $Y = \bigcap_{b \in S} U(b)$ *be a subspace of* Y*,* $S \subset G$*. If* $P(\underline{a})$ *fails in* Y *but holds in every finite subspace of* Y*, then, for some* $\{b_1, \ldots, b_l\} \subset S$*,* $P_l(\underline{b}, \underline{a})$ *fails in* (X, G) *but holds in every finite subspace of* (X, G).

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