RESULTANTS AND THE BEZOUT THEOREM

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1. The Intersections of Plane Curves

Let K be an algebraically closed field. Consider polynomials $f, g \in K[X]$:

$$\begin{array}{rcll} f(X) & = & a_0X^s + a_1X^{s-1} + \ldots + a_s, & a_0 \neq 0, \\ g(X) & = & b_0X^t + g_1X^{t-1} + \ldots + b_t, & b_0 \neq 0. \end{array} \tag{1}$$

We define the resultant of polynomials (1) by:

$$Res(f,g) = \begin{vmatrix} a_0 & a_1 & \ldots & \ldots & \ldots & a_s & 0 & \ldots & 0 \\ 0 & a_0 & \ldots & \ldots & \ldots & a_{s-1} & a_s & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & a_0 & a_1 & \ldots & \ldots & a_s \\ b_0 & b_1 & \ldots & \ldots & b_t & 0 & \ldots & \ldots & 0 \\ 0 & b_0 & \ldots & \ldots & b_{t-1} & b_t & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & b_0 & b_1 & \ldots & \ldots & \ldots & b_t \end{vmatrix}.$$

During the class we have already shown the following theorem:
Corollary 6.15 $gcd(f,g) = 1$ iff. $Res(f,g) \neq 0$.

Now we would like to generalize this result to the case of the polynomials in two variables and use it to count points of intersections of plane curves. Firstly we have to show some basic properities of resultants.

Lemma 1. *Under the above notation $Res(f,g) = (-1)^{st}Res(g,f)$.*

Proof. In order to obtain $Res(g,f)$ from $Res(f,g)$ we shall interchange $t\dot{s}$ rows in the determinant. $\square$

Lemma 2. *Under the above notation there exist polynomials $p, q \in K[X]$ such that:*

$$pf + gq = Res(f,g).$$

Proof. Observe that:

$$Res(f,g) = \begin{vmatrix} a_0 & a_1 & \dots & a_s & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_0 & a_1 & \dots & a_s \\ b_0 & b_1 & \dots & b_t & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_0 & b_1 & \dots & b_t \end{vmatrix} =$$

$$= \begin{vmatrix} a_0 & \dots & a_s & 0 & \dots & a_0X^{s+t-1} + a_1X^{s+t-2} + \dots + a_sX^{t-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & a_0 & a_1 & \dots & a_0X^t + a_1X^{t-1} + \dots + a_s \\ b_0 & \dots & b_t & 0 & \dots & b_0X^{s+t-1} + b_1X^{s+t-2} + \dots + b_tX^{s-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & b_0 & b_1 & \dots & b_0X^s + a_1X^{s-1} + \dots + b_t \end{vmatrix} =$$

$$= \begin{vmatrix} a_0 & a_1 & \dots & a_s & 0 & \dots & X^{t-1}f \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_0 & a_1 & \dots & f \\ b_0 & b_1 & \dots & b_t & 0 & \dots & X^{s-1}g \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_0 & b_1 & \dots & g \end{vmatrix}.$$

We finish the proof by expanding the last determinant by the last column and putting together terms with f and g □

Lemma 3. *Under the above notation let $Res_0(f,g)$ be the determinant obtained from $Res(f,g)$ by setting:*

$$b_0 = b_1 = \dots = b_{k-1} = 0, \quad k \le t.$$

Let $g_0(X) = b_kX^{t-k} + \dots + b_t$, $b_k \neq 0$. Then:

$$a_0^k Res(f, g_0) = Res_0(f,g).$$

Proof. Follows immediately from expansion $Res_0(f,g)$ along the first column. □

Lemma 4. *Under the above notation let $Res_1(f,g)$ be the determinant obtained from $Res(f,g)$ by setting:*

$$a_0 = a_1 = \dots = a_{l-1} = 0, \quad l \le s.$$

Let $f_1(X) = a_lX^{s-l} + \dots + a_s$, $a_s \neq 0$. Then:

$$(-1)^{lt} b_0^l Res(f_1, g) = Res_1(f,g).$$

Proof. Follows immediately from lemmas 1 and 3. □

In all these considerations the field K can be replaced by the field of rational functions, and $a_0, \dots, a_s, b_0, \dots, b_t$ may be polynomials. Consider polynomials $f, g \in K[X,Y]$:

$$\begin{aligned} f(X,Y) &= f_0(X)Y^s + f_1(X)Y^{s-1} + \dots + f_s(X), \\ g(X,Y) &= g_0(X)Y^t + g_1(X)Y^{t-1} + \dots + g_t(X). \end{aligned} \tag{2}$$

Define the resultant of polynomials (2) by:

$$Res_Y(f,g)(X) =$$

$$= \begin{vmatrix} f_0(X) & \dots & \dots & \dots & f_s(X) & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & f_{s-1}(X) & f_s(X) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & f_0(X) & f_1(X) & \dots & \dots & f_s(X) \\ g_0(X) & \dots & \dots & g_t(X) & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & g_{t-1}(X) & g_t(X) & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & g_0(X) & g_1(X) & \dots & \dots & \dots & g_t(X) \end{vmatrix}.$$

We have the following generalization of Corollary 6.15:

Theorem 1. *$gcd_Y(f,g) \in K[X]$ iff. $Res_Y(f,g)(X) \neq const.0$.*

Proof. Follows immediately from Corollary 6.15. □

We will now apply this theorem for solving systems of algebraic equations. Consider the system:

$$\left\{ \begin{array}{rcl} f(x,y) & = & 0 \\ g(x,y) & = & 0. \end{array} \right. \tag{3}$$

We shall prove:

Theorem 2. *There exists a solution (a,b) of the system (3), $(f_0(a), g_0(a)) \neq (0,0)$ iff. $Res_Y(f,g)(a) = 0$, $(f_0(a), g_0(a)) \neq (0,0)$.*

Proof. ($\Rightarrow$) Suppose $f(a,b) = 0$, $g(a,b) = 0$. Let:

$$f_a(Y) = f_0(a)Y^s + \ldots + f_s(a) \in K[Y], \quad g_a(Y) = g_0(a)Y^t + \ldots + g_t(a) \in K[Y].$$

Since $f_a(b) = g_a(b) = 0$, $f_a, g_a \in K[Y]$ have the common root. If $f_0(a) \neq 0$ and $g_0(a) \neq 0$, then by Corollary 6.15 $Res_Y(f,g)(a) = Res(f_a, g_a) = 0$. If $f_0(a) \neq 0$ or $g_0(a) \neq 0$, then we may assume $f_0(a) \neq 0$. Say $g_0(a) = g_1(a) = \ldots = g_{k-1}(a) = 0$, $g_k(a) \neq 0$, $k \leq t$. Denote by $Res_0(f_a, g_a)$ the determinant obtained from $Res(f_a, g_a)$ by substituting $g_0(a) = g_1(a) = \ldots = g_{k-1}(a) = 0$ and let $g_{0a}(Y) = g_k(a)Y^{t-k} + \ldots + g_f(a) \in K[Y]$. Of course $g_{0a}(b) = g_a(b)$, so $f_a, g_{0a} \in K[Y]$ have the common root. By Lemma 3 and Corollary 6.15:

$$Res_Y(f,g)(a) = Res_1(f_a, g_a) = (f_0(a))^k Res(f_a, g_{0a}) = 0.$$

($\Leftarrow$) Suppose $Res_Y(f,g)(a) = 0$. Let:

$$f_a(Y) = f_0(a)Y^s + \ldots + f_s(a) \in K[Y], \quad g_a(Y) = g_0(a)Y^t + \ldots + g_t(a) \in K[Y].$$

If $f_0(a) \neq 0$ and $g_0(a) \neq 0$, then $0 = Res_Y(f,g)(a) = Res(f_a, g_a)$ and by Corollary 6.15 f_a and g_a have common factor, which - since K is algebraically closed - has a root b. If $f_0(a) \neq 0$ or $g_0(a) \neq 0$, then we may assume $f_0(a) \neq 0$. Say $g_0(a) = g_1(a) = \ldots = g_{k-1}(a) = 0$, $g_k(a) \neq 0$, $k \leq t$. Denote by $Res_0(f_a, g_a)$ the determinant obtained from $Res(f_a, g_a)$ by substituting $g_0(a) = g_1(a) = \ldots = g_{k-1}(a) = 0$ and let $g_{0a}(Y) = g_k(a)Y^{t-k} + \ldots + g_f(a) \in K[Y]$. By Lemma 3:

$$0 = Res_Y(f,g)(a) = Res_1(f_a, g_a) = (f_0(a))^k Res(f_a, g_{0a}),$$

so $Res(f_a, g_{0a}) = 0$. Therefore, by Corollary 6.15, f_a and g_a have common factor and hence a common root b. Of course $g_{0a}(b) = g_a(b) = g(a,b)$, which finishes the proof. □

Theorem 2 implies the following method of solving the system (3):

- Form the resultant $Res_Y(f,g)(X) \in K[X]$.
- Each root of $Res_Y(f,g)(X)$ which does not satisfy the equations $f_0(x) = 0$, $g_0(x) = 0$ is an "a" from at least one solution (a,b).
- Form the resultant $Res_X(f,g)(Y) \in K[Y]$.
- Each root of $Res_X(f,g)(Y)$ which does not satisfy the equations $f_0(y) = 0$, $g_0(y) = 0$ is a "b" from at least one solution (a,b).

Let, for example:

$$f(X,Y) = X^2 - 2XY + 3X, \quad g(X,Y) = Y^2 - 4X.$$

We have:

$$\begin{aligned} Res_Y(f,g)(X) &= \begin{vmatrix} -2X & X^2+3 & 0 \\ 0 & -2X & X^2+3 \\ 1 & 0 & -4X \end{vmatrix} = X^2(X^2 - 10X + 9), \\ Res_X(f,g)(Y) &= \begin{vmatrix} 1 & 3-2Y & 0 \\ -4 & Y^2 & 0 \\ 0 & -4 & Y^2 \end{vmatrix} = Y^2(Y^2 - 8Y + 12). \end{aligned}$$

Thus the possible "a's" are 0, 1 and 9 and possible "b's" are 0, 2, 6. Hence only the following pairs can be solutions of (3):

$$(0,0),(0,2),(0,6),(1,0),(1,2),(1,6),(9,0),(9,1),(9,6).$$

We can easy find that only $(0,0),(1,2),(9,6)$ are solutions of the system (3).

2. The Basic Version of the Bezout Theorem

Consider the following example:

$$f(X,Y) = XY^2 - Y + X^2 + 1, \quad g(X,Y) = X^2Y^2 + Y - 1. \tag{4}$$

An easy calculation leads to the result:

$$\begin{aligned} Res_Y(f,g)(X) &= X^2(X^6 + 2X^4 + 2X^3 + 2X^2 + 3X + 1), \\ Res_X(f,g)(Y) &= (Y-1)(Y^6 + Y^5 - Y^4 + 2Y^3 - 2Y^2 + Y - 1). \end{aligned}$$

In particular, $Res_Y(f,g)(0) = 0$. But on the other hand, since $f_0(X) = X$, $g_0(X) = X^2$, we have:

$$f_0(0) = g_0(0) = 0,$$

so the points of the type $(0,b)$ do not satisfy the hypoteses of Theorem 2. However, it is easy to verify that - for example - the point $(0,1)$ is the solution of (4). Therefore - in order to solve the system of the form (3) - we need the upper bound for the number of solutions. The so called **Bezout theorem** (or, at least, a version of it) gives such bound:

Theorem 3 (Bezout). *Let $f, g \in K[X,Y]$ be the polynomials of degrees r and s, respectively. If f and g have no common factor of degree > 0, then there exist at most $r\dot{s}$ solutions of system (3).*

In order to prove the Bezout theorem, we will need some general properities of resultants.

Lemma 5. *Let $f, g \in K[X_1, \dots, X_r, Y_1, \dots, Y_s, T] \subset K(X_1, \dots, X_r, Y_1, \dots, Y_s)[T]$ be such that:*

$$\begin{aligned} f(T) &= a_0 T^r + \ldots + a_r \in K(X_1, \dots, X_r, Y_1, \dots, Y_s)[T], \\ g(T) &= b_0 T^s + \ldots + b_s \in K(X_1, \dots, X_r, Y_1, \dots, Y_s)[T], \end{aligned}$$

where $a_j \in K[X_1, \dots, X_r]$, $dega_j = j$, $b_j \in K[Y_1, \dots, Y_s], degb_j = j$ are monic polynomials. Let $a_0 = \ldots = a_{k-1} = 0$, $a_k \neq 0$, $b_0 = \ldots = b_{m-1} = 0$, $b_m \neq 0$. Then $Res(f, g) \in K(X_1, \dots, X_r, Y_1, \dots, Y_s)$ is a monic polynomial of degree $rs - km$.

Proof. Obviously $R(f, g) \in K[X_1, \dots, X_r, Y_1, \dots, Y_s]$. Fix an element $c \in K$ and $(x_1, \dots, x_r, y_1, \dots, y_s) \in K^{r+s}$. We will show that:

$$R(f,g)(cx_1, \dots, cx_r, cy_1, \dots, cy_s) = c^{rj-km} R(f,g)(x_1, \dots, x_r, y_1, \dots, y_s).$$

Indeed, we have:

$$R(f,g)(cx_1, \dots, cx_r, cy_1, \dots, cy_s) =$$

$$= \begin{vmatrix} c^k a_k & c^{k+1} a_{k+1} & \dots & c^r a_r & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c^k a_k & c^{k+1} a_{k+1} & \dots & c^r a_r \\ c^m b_m & c^{m+1} b_{m+1} & \dots & c^s b_s & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c^m b_m & c^{m+1} b_{m+1} & \dots & c^s b_s \end{vmatrix} \begin{array}{l} \leftarrow \cdot c^m \\ \vdots \\ \leftarrow \cdot c^{s-1} \\ \leftarrow \cdot c^k \\ \vdots \\ \leftarrow \cdot c^{r-1} \end{array} =$$

$$= \frac{1}{c^{\frac{m+s-1}{2}(m-s) + \frac{k+r-1}{2}(k-r)}} \cdot$$

$$\cdot \begin{vmatrix} c^{k+m} a_k & c^{k+m+1} a_{k+1} & \dots & c^{r+m} a_r & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c^{k+s-1} a_k & c^{k+s} a_{k+1} & \dots & c^{r+s-1} a_r \\ c^{k+m} b_m & c^{k+m+1} b_{m+1} & \dots & c^{k+s} b_s & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c^{m+r-1} b_m & c^{m+r} b_{m+1} & \dots & c^{r+s-1} b_s \end{vmatrix}.$$

Thus:

$$\begin{aligned} & c^{\frac{m+s-1}{2}(m-s) + \frac{k+r-1}{2}(k-r)} R(f,g)(cx_1, \dots, cy_s) = \\ &= c^{(k+m)+(k+m+1)+\ldots+(r+s-1)} Res(f,g)(x_1, \dots, y_s), \end{aligned}$$

so by comparing the powers of c we finish the proof. $\square$

Lemma 6. *Let $f, g \in K(X_1, \dots, X_r, Y_1, \dots, Y_s)[T]$ be such that:*

$$\begin{aligned} f(T) &= (T - X_1)(T - X_2) \dots (T - X_r), \\ g(T) &= (T - Y_1)(T - Y_2) \dots (T - Y_s). \end{aligned}$$

Then $Res(f, g) = \prod_{j=1}^{r} \prod_{k=1}^{s} (X_j - Y_k)$.

Proof. By the Viete formulae:

$$\begin{aligned} f(T) &= T^r - S_1(X_1, \dots, X_r) T^{r-1} + \ldots + (-1)^r S_r(X_1, \dots, X_r), \\ g(T) &= T^s - S_1(Y_1, \dots, Y_s) T^{s-1} + \ldots + (-1)^s S_s(Y_1, \dots, Y_s), \end{aligned}$$

where S_i are the primitive symmetric polynomials. Of course each S_i is a non-zero monic polynomial in r or s variables, so by Lemma 5 $Res(f, g)$ is a monic polynomial of degree rs. By the Corollary 6.15 $Res(f, g) = 0$ when $X_j = Y_k$ for some j and

k. Therefore, as $Res(f,g) \in K(X_1,\dots,X_{j-1},X_{j+1},\dots,X_r,Y_1,\dots,Y_s)[X_j]$ and $Res(f,g)(Y_k)=0$, we have that $X_j - Y_k | Res(f,g)$. Since X_j and Y_k are algebraically independent, $X_j - Y_k$ and $X_l - Y_p$ are relatively prime if $j \neq l$ or $k \neq p$. Thus $Res(f,g)$ is divisible by the product of all $X_j - Y_k$:

$$Res(f,g) = \lambda \prod_{j=1}^{s} (X_j - Y_k).$$

Since $deg \prod_{j=1}^{r} \prod_{k=1}^{s} (X_j - Y_k) = rs = deg Res(f,g)$, we have that λ is constant.

Now we only have to show that $\lambda = 1$. Indeed, set $X_1 = \ldots = X_r = 0$, $Y_k = \cos(\frac{2\pi k}{s}) + i\sin(\frac{2\pi k}{s})$. Then $f(T) = T^r$, $g(T) = T^s - 1$ and it is easy to verify that $Res(f,g) = (-1)^r$. Since $\prod_{k=1}^{s}(T - Y_k) = T^s - 1$, we have $\prod_{j=1}^{r}\prod_{k=1}^{s} = \prod_{j=1}^{r}(X_j^s - 1) = (-1)^r$. As λ is independent on the values of X_j, Y_k, this finishes the proof. □

Lemma 7. *Let $f, g \in K[T]$ be such that:*

$$\begin{aligned} f(T) &= a_0T^r + \ldots + a_r = a_0(T - c_1)\ldots(T - c_r), \\ g(T) &= b_0T^s + \ldots + b_s = b_0(T - d_1)\ldots(T - d_s). \end{aligned}$$

Then:

$$Res(f,g) = a_0^s b_0^r \prod_{j=1}^{r}\prod_{k=1}^{s}(c_j - d_k) = a_0^s \prod_{j=1}^{r} g(c_j) = (-1)^{rs} b_0^s \prod_{k=1}^{s} f(d_k).$$

Proof. If $a_0 = b_0 = 1$ then, by substitution $X_j = c_j$, $Y_k = d_k$ in Lemma 6 we have:

$$Res(f,g) = \prod_{j=1}^{r}\prod_{k=1}^{s}(c_j - d_k). \tag{5}$$

If $a_0 \neq 1$, $b_0 \neq 1$, then the resultant $Res(f,g)$ is obtained from $Res(\frac{f}{a_0}, \frac{g}{b_0})$ by multiplying first s rows by a_0 and last r rows by b_0. Hence:

$$Res(f,g) = a_0^s b_0^r \prod_{j=1}^{r}\prod_{k=1}^{s}(c_j - d_k)$$

follows directly from (5). To prove the rest of the theorem observe that since:

$$g(T) = b_0 \prod_{k=1}^{s}(T - d_k),$$

we have:

$$a_0^s prod_{j=1}^{r} g(c_j) = a_0^s \prod_{j=1}^{r}(b_0 \prod_{k=1}^{s}(c_j - d_k)) = a_0^s b_0^r \prod_{j=1}^{r}\prod_{k=1}^{s}(c_j - d_k).$$

Similarly:

$$(-1)^{rs} b_0^s \prod_{k=1}^{s} f(d_k) = a_0^s b_0^r \prod_{j=1}^{r}\prod_{k=1}^{s}(c_j - d_k).$$

□

Define the **discriminant** of the polynomial f of degree r as the number:

$$\Delta(f) = (-1)^{\frac{r(r-1)}{2}} \frac{1}{a_0} Res(f, f').$$

The next lemma gives us an useful property of discriminant:

Lemma 8. *Let $f \in K[T]$ be such that:*

$$f(T) = a_0 T^r + \ldots + a_r = a_0(T - c_1)\ldots(T - c_r).$$

Then:

$$\Delta(f) = a_0^{2r-2} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ c_1 & c_2 & \ldots & c_r \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{r-1} & c_2^{r-1} & \ldots & c_r^{r-1} \end{vmatrix}^2 .$$

Proof. We have:

$$f'(T) = a_0 \sum_{j=1}^{r} (T - c_1)\ldots(T - c_{j-1})(T - c_{j+1})\ldots(T - c_r).$$

Hence it follows:

$$\begin{aligned} f'(c_j) &= a_0(c_j - c_1)\ldots(c_j - c_{j-1})(c_j - c_{j+1})\ldots(c_j - c_r) = \\ &= (-1)^{r-j} a_0(c_j - c_1)\ldots(c_j - c_{j-1})(c_{j+1} - c_j)\ldots(c_r - c_j) \end{aligned}$$

for $j \in \{1, \ldots, r\}$. By Lemma 7:

$$\begin{aligned} & Res(f, f') = a_0^{r-1} \prod_{j=1}^{r} f'(c_j) = \\ = \; & a_0^{2r-1}(-1)^{(r-1)+(r-2)+\ldots+1+0}(c_2 - c_1)(c_3 - c_1)\ldots(c_r - c_1)\cdot \\ & \cdot(c_2 - c_1)(c_3 - c_2)\ldots(c_r - c_2)(c_r - c_1)(c_r - c_2)\ldots(c_r - c_{r-1}). \end{aligned}$$

In the above product every difference $c_k - c_j$ occurs twice. Thus:

$$a_0 \Delta(f) = (-1)^{\frac{r(r-1)}{2}} Res(f, f') = a_0^{2r-1} \prod_{k>j} (c_k - c_j)^2.$$

□

Finally we can prove Bezout theorem. The main idea is based on the following lemma:

Lemma 9. *Let $f, g \in K[X, Y]$ be the polynomials of degrees r and s respectively. Then $degRes(f, g) \leq rs$.*

Proof. Substitute $X := \frac{X_1}{X_2}$, $Y := \frac{Y}{X_2}$. Then $f, g \in K(X_1, X_2, Y)$. We find the common denominator and then cancel it. Thus we obtain two monic polynomials $f^+, g^+ \in K[X_1, X_2, Y]$ of degrees r and s. Write:

$$\begin{aligned} f^+(X_1, X_2, Y) &= f_0(X_1, X_2)Y^r + \ldots + f_r(X_1, X_2), \\ g^+(X_1, X_2, Y) &= g_0(X_1, X_2)Y^s + \ldots + g_s(X_1, X_2), \end{aligned}$$

where $f_j, g_k \in K[X_1, X_2]$ are monic polynomials either equal to 0 or of degrees j nad k, respectively. By Lemma 5 $Res(f^+, g^+)$ (with respect to Y) is a monic polynomial of degree $\leq rs$. Since $f(X_1, Y) = f^+(X_1, 1, Y)$ and $g(X_1, Y) = g^+(X, 1, Y)$ and

$Res_Y(f,g)$ is obtained from $Res(f^+,g^+)$ by substituting $X_2=1$, this proves the lemma. $\square$

Now suppose that $f,g\in K[X,Y]$ are the polynomials of degrees r and s, respectively, having no commong factor of degree >0 and that there exist $rs+1$ different solutions of the system (2). Observe that there exists a number s such that:

$$a_j+cb_j\neq a_k+cb_k,\quad 1\leq j<k\leq rs+1$$

(elsewhere, substituting $c=0$ and $c=1$ we would obtain $a_j=a_k$ and $b_j=b_k$ for some j and k). Substitute:

$$X:=X'-cY',\quad Y=Y'$$

and let:

$$\begin{aligned} f(X,Y) &= f_0(X)Y^r+f_1(X)Y^{r-1}+\ldots+f_r(X),\\ g(X,Y) &= g_0(X)Y^s+g_1(X)Y^{s-1}+\ldots+g_s(X). \end{aligned}$$

Then:

$$\begin{aligned} f_1(X',Y') &= f_0(X'-cY')Y'^r+f_1(X'-cY')Y'^{r-1}+\ldots+f_r(X'-cY'),\\ g_1(X',Y') &= g_0(X'-cY')Y'^s+g_1(X'-cY')Y^{s-1}+\ldots+g_s(X'-cY') \end{aligned}$$

are the polynomials of degrees r and s, respectively. Set:

$$(a'_j,b'_j)=(a_j+cb_j,b_j),\quad j\in\{1,2,\ldots,rs=1\}.$$

We have $a'_j\neq a'_k$ for $j\neq k$ and it is easy to verify, that:

$$f_1(a'_j,b'_j)=g_1(a'_j,b'_j)=0.$$

By Theorem 2, $a'_1,a'_2,\ldots,a'_{rs+1}$ are roots of the resultant $Res_{Y'}(f_1,g_1)(X')$ of polynomials f_1 and g_1. By Lemma 9, $degRes_{Y'}(f_1,g_1)(X')\leq rs$, so $Res_{Y'}(f_1,g_1)(X')$ is the zero-polynomial. By Theorem 1 $gcd_Y(f,g)\notin K[X]$, which is a contradiction to the assumptions of the theorem. Therefore the Bezout theorem is proved.

3. Statement of the General Version of the Bezout Theorem

Let $f(X,Y)\in F[X,Y]$ be a polynomial. An **affine plane algebraic curve** is the set:

$$\mathcal{Z}(f)=\{(x,y):f(x,y)=0\}.$$

Our aim is to figure out missing assumptions of the following theorem:

Theorem 4 (Bezout)**.** *Two plane $\square_1$ algebraic curves $\mathcal{Z}(f)$, $\mathcal{Z}(g)$ $\square_2$ of degrees $d=deg(f)$ and $e=deg(g)$ over a field $\square_3$ F have $\square_4$ $d\cdot e$ common points.*

We write $\square_i$ to denote the missing assumption. It is easy to complete assumption $\square_3$. Consider curves:

$$\begin{aligned} f(X,Y) &= Y,\ (OX \text{ axis, degree } 1)\\ g(X,Y) &= Y-h(X),\ (\text{a graph of polynomial } h(X)\in F[X], \text{ degree } degh(X)). \end{aligned}$$

It follows from the Bezout theorem, that every polynomial of degree n has n roots in F. Thus:

$$\square_3=\text{"algebraically closed"}.$$

Moreover, consider curves:

$$\begin{aligned} f(X,Y) &= Y, \\ g(X,Y) &= Y - X^2. \end{aligned}$$

As we expect that f nad g have 2 points of intersection, we must assume:

$$\Box_4 = \text{"counted with multiplicities"}.$$

It is clear that if $gcd(f,g) \neq 1$, then $\mathcal{Z}(f)$, $\mathcal{Z}(g)$ mave infinitely many points of intersection, hence:

$$\Box_2 = \text{"without common factor"}.$$

Finally, consider two lines:

$$\begin{aligned} f(X,Y) &= Y, \\ g(X,Y) &= Y + 1. \end{aligned}$$

If we want $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ to intersect, we must - instead of "affine" geometry - go to "projective" geometry. Thus:

$$\Box_1 = \text{"projective"}.$$

Hypoteses $\Box_3$ and $\Box_2$ are clear. However we should define what do "projective curve" and "multiplicities of intersection" mean. The first definition is quite intuitive - we need an polynomial equation whose zeroes are homogenous triples $(a : b : c)$. The monic polynomials are good candidates. Definition of multiplicity of intersection is far from clear and can not be explained in this short project. The reader should refer to Hartshorne's "Algebraic geometry" for more details.

References

[1] J. von zur Gathen, J. Gerhard, *Modern Computer Algebra*
[2] A. Mostowski, M. Stark, *Introduction to Higher Algebra*, Pergamon Press, 1964
[3] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, 2001