1. Krull valuations

Let G be an additive and commutative group. A subset $S \subsetneq G$ is said to be an **ordering** of the group G if:

(1) $\bigwedge_{s_1,s_2\in S} s_1 + s_2 \in S$,
(2) $\bigwedge_{a\in G} a \in S \vee -a \in S$,
(3) $S \cap -S = \{0\}$, where $-S = \{a \in G : -a \in S\}$.

For the given ordering S of the group G we denote:

$$a \leq_S b \Leftrightarrow b - a \in S.$$

It is easy to prove that the relation $\leq \subset G \times G$ is a linear ordering, that is:

(1) $\bigwedge_{a\in G} a \geq a$,
(2) $\bigwedge_{a,b\in G} a \geq b \wedge b \geq a \Rightarrow a = b$,
(3) $\bigwedge_{a,b,c\in G} a \geq b \wedge b \geq c \Rightarrow a \geq c$,
(4) $\bigwedge_{a,b\in G} a \geq b \vee b \geq a \vee a = b$,

such that

(5) $\bigwedge_{a,b,c\in G} a \geq b \Rightarrow a + c \geq b + c$,

if and only if the set $S = \{a \in G : a \geq 0\}$ is an ordering. Moreover, observe that ordered abelian groups do not have elements of finite order: indeed, suppose that $a \in G$ is an element of order n. We may assume that $a > 0$. Then $0 = na = a + a + \ldots + a > 0$, which is a contradiction. For the ordered group G we define a **projective group** $G \cup \{\infty\}$ consistent of the group G with its ordering and a symbol ∞ which satisfies the following conditions:

(1) $\bigwedge_{a\in G} a < \infty$,
(2) $\bigwedge_{a\in G} a + \infty = \infty + a = \infty$.

Now we can define a **Krull valuation**. Let F be a field and $G \cup \{\infty\}$ an ordered projective group called the **value group**. A function $v : F \to G \cup \{\infty\}$ is said to be the Krull valuation (or simply the valuation) when:

(1) $\bigwedge_{a\in F} v(a) = \infty \Leftrightarrow a = 0$,
(2) $\bigwedge_{a,b\in F} v(ab) = v(a) + v(b)$,
(3) $\bigwedge_{a,b\in F} v(a + b) \geq \min\{v(a), v(b)\}$, provided $a + b \neq 0$.

Observe that if $v : F \to G \cup \{\infty\}$ is the valuation, then:

$$v(1) = 0.$$

Indeed, we have that $v(1) = v(1 \cdot 1) = v(1) + v(1)$, hence $v(1) = 0$. Similarly, using the identity $a \cdot a^{-1} = 1$ we can show that:

$$v(a^{-1}) = -v(a).$$

Next, since $0 = v(1) = v((-1) \cdot (-1)) = v(-1) + v(-1)$, we have that $v(-1) = 0$ or $v(-1)$ is an element of order 2. The second possibility cannot hold, so $v(-1) = 0$ and thus:

$$v(-a) = v(a).$$

Finally, observe that:

$$v(a) \neq v(b) \Rightarrow v(a + b) = \min\{v(a), v(b)\}.$$

Indeed, suppose that for some $a, b \in F$ we have $v(a) \neq v(b)$. We may assume that $v(a) < v(b)$. Suppose that $v(a+b) \neq \min\{v(a), v(b)\}$. This implies $v(a+b) > \min\{v(a), v(b)\}$, in particular $v(a+b) > v(a)$. Thus:

$$v(a) = v((a+b) - b) \geq \min\{v(a+b), v(b)\} > v(a)$$

which is a contradiction.

Next we shall introduce the notion of **valuation rings**. Let F be a field. A ring $A \subset F$ is called the **valuation ring** if:

$$\bigwedge_{a \in F} a \in A \vee a^{-1} \in A.$$

If the field F is not given we shall assume that F is the field of fractions of A. There is a natural correspondence between valuations and valuation rings which is described by the two theorems that we shall state.

Theorem 1. *Let $v : F \to G \cup \{\infty\}$ be a valuation.*

(1) *The set:*
$$A_v = \{a \in F : v(a) \geq 0\}$$
is a valuation ring. We shall call it the **valuation ring associated with v.**

(2) *The set:*
$$M_v = \{a \in F : v(a) > 0\}$$
is the only maximal ideal in the ring A_v. In particular, A_v is a local ring and $F_v = A_v/M_v$ is a field, which shall be called the bf residue field of v.

(3) *The set:*
$$U_v = \{a \in F : v(a) = 0\}$$
is a group consistent of all units of the ring A_v.

Proof. (1) By the definition of a valuation we can directly check that A_v is a ring. To show that this is a valuation ring fix $a \in F$ and suppose that $a \notin A_v$. Then $v(a) < 0$. This implies that $v(a^{-1}) = -v(a) > 0$, so $a \in A_v$.

(2). Without any difficulty we can verify that M_v is an ideal. In order to check that this is the only maximal ideal it suffices to show that $a \in M_v$ if and only if $a \notin U(A_v)$. Suppose that $a \in M_v$, that is $v(a) > 0$. Suppose that there exists $b \in A_v$ such that $a \cdot b = 1$. Then $v(b) \geq 0$, but also $v(b) = v(a^{-1}) = -v(a) < 0$ - a contradiction. Conversely, suppose that for a fixed $a \in A_v$ we have $a \notin M_v$, that is $v(a) = 0$. Since F is a field, for some $b \in F$ we have $a \cdot b = 1$. Then $0 = v(1) = v(ab) = v(a) + v(b) = v(b)$, so $b \in A_v$. That means that $a \in U(A_v)$. (3) follows immediately from (2). $\square$

This theorem states that for every valuation v we can choose a valuation ring A_v. We will prove that the converse is also true, that is that for any valuation ring A there is the cannonical valuation v_A, whose valuation ring is the same as A. We need some preliminaries, though. For a given field F we define a **divisibility** relation $| \subset F \times F$ such that:

(1) $\bigwedge_{a \in F} a|a$,
(2) $\bigwedge_{a,b,c \in F} a|b \wedge b|c \Rightarrow a|c$,
(3) $\bigwedge_{a,b,c \in F} a|b \Rightarrow ac|bc$,
(4) $\bigwedge_{a,b,c \in F} a|b \wedge a|c \Rightarrow a|b - c$.

Clearly for a given divisibility relation $|$ in F:

$$\bigwedge_{a\in F} a|0 \wedge \sim 0|a.$$

Moreover, the family of all divisibility relations in the field F is in bijective correspondence with the family of all subrings of F. More precisely, if $R \subset F$ is a subring, then the condition:

$$a|b \Leftrightarrow b\cdot a^{-1} \in R$$

defines a divisibility relation and for the given divisibility relation $| \subset F\times F$ the set:

$$R = \{a \in F : 1|a\}$$

is a subring of F with the following group of units:

$$U(R) = \{a \in R : a|1\}.$$

It is easy to see that R is a valuation ring if and only if the corresponding divisibility relation $|$ is total, that is:

$$\bigwedge_{a,b\in F} a|b \vee b|a.$$

Assume that $|$ is total and fix $a \in F$ such that $a \notin R$. Then $\sim 1|a$ (otherwise $a\cdot 1^{-1} = a \in R$) and since $|$ is total it follows that $a|1$. Hence $a^{-1} = 1\cdot a^{-} \in R$. Conversely, suppose that R is a valuation ring and fix $a, b \in F$ such that $\sim a|b$. If $a = 0$ then obviously $b|a$, so assume that $a \neq 0$. Then $\sim 1|ba^{-1}$ and hence $ba^{-1} \notin R$. But R is a valuation ring, so $ab^{-1} = (ba^{-1})^{-1} \in R$. Thus $1|ab^{-1}$, that is $b|a$. Now we are able to state the next theorem:

Theorem 2. *Let A be a valuation ring in F. There exists a Krull valuation $v_A : F \to G_A \cup \{\infty\}$ such that:*

$$A_{v_A} = A\ ,\quad M_{v_A} = A\setminus U(A)\ ,\quad U_{v_A} = U(A).$$

Proof. Fix a valuation ring A and let $|$ be the corresponding divisibility relation given by

$$a|b \Leftrightarrow ba^{-1} \in A.$$

By the previous remark such relation is total. Consider the quotient additive group $G_A = U(F)/U(A)$. Define the relation $\leq \subset G_A \times G_A$ by:

$$a + U(A) \leq b + U(A) \Leftrightarrow a|b.$$

Since $|$ is reflexive, transitive, total and it agrees qith multiplication, the relation $\leq$ is also reflexive, transitive, total and it agrees with addition. It is also antisymmetric - suppose that $a + U(A) \leq b + U(A)$ and $b + U(A) \leq a + U(A)$. Thus $a|b$ and $b|a$, so $1|ba^{-1}$ and $ba^{-1}|1$. Therefore $ba^{-1} \in A$ and $ba^{-1} \in U(A)$, which means that $a + U(A) = b + U(A)$. Thus the group G_A is an ordered abelian group.

Define the mapping $v_A : F \to G_A \cup \{\infty\}$ by

$$v_A(a) = \begin{cases} a + U(A), & \text{gdy } a \neq 0,\\ \infty, & \text{gdy } a = 0.\end{cases}$$

We shall show that v_A is a valuation which will be called the **canonical valuation.** Obviously $A_{v_A} = A$, since $a \in A_v$ if and only if $v_A(a) = a + U(A) \geq 1 + U(A) = U(A)$, that is when $1|a$ - or in other words when $a \in A$. Similarly we can check that $M_{v_A} = A \setminus U(A)$ and $U_{v_A} = U(A)$. It is clear that v_A is surjective and $v_A(a) = \infty$ if and only if $a = 0$. Since v_A is a homomorphism from a multiplicative group into an additive group, we have $v_A(ab) = v_A(a) + v_A(b)$. It remains to show that $v_A(a+b) \geq \min\{v_A(a), v_A(b)\}$.

Fix $a, b \in F$. We may assume that $v_A(a) \leq v_A(b)$. That means that $a + U(A) \leq b + U(A)$, that is $a|b$ - hence $ba^{-1} \in A$. Since A is a commutative ring with identity, we get $1 + ba^{-1} \in A$. Thus $1|1 + ba^{-1}$, so $U(A) = 1 + U(A) \leq 1 + ba^{-1} + U(A)$. Finally:

$$\begin{aligned} v_A(a+b) &= (a+b) + U(A) = a(1 + ba^{-1}) + U(A) = \\ &= (a + U(A)) + ((1 + ba^{-1}) + U(A)) \geq \\ &\geq a + U(A) = v_A(a) = \min\{v_A(a), v_A(b)\}. \end{aligned}$$

□

The previous two theorems in the fact establish an almost bijective relation between valuations and valuation rings. If $v_1 : F \to G_1 \cup \{\infty\}$ and $v_2 : F \to G_2 \cup \{\infty\}$ are two valuations, then we say that they are **equivalent**, written $v_1 \simeq v_2$, if there exists an order preserving group isomorphism $g : G_1 \to G_2$ such that $v_2 = g \circ v_1$ (we take $g(\infty) = \infty$). Clearly such relation is an equivalence and we can state the following result:

Theorem 3. *The set of all equivalence classes of the relation $\simeq$ is in a bijective correspondence with the family of all valuation rings in F.*

Proof. Suppose that $v_1 : F \to G_1 \cup \{\infty\}$ and $v_2 : F \to G_2 \cup \{\infty\}$ are equivalent. Then $v_1(a) \geq 0$ if and only if $v_2(a) \geq 0$, so $A_{v_1} = A_{v_2}$.

Conversely, let A be a valuation ring and let $A = A_v$ for some valuation $v : F \to G \cup \{\infty\}$. By the previous theorem $A = A_{v_A}$ for the cannonical valuation $v_A : F \to G_A \cup \{\infty\}$. We shall show that $v \approx v_A$. Observe that $v|_{U(F)} : U(F) \to G$ is a surjective homomorphism and that $\ker v|_{U(F)} = U(A)$. By the isomorphism theorem $G_A = U(F)/U(A) \simeq G$. If $g : G_A \to G$ is such isomorphism, then it is easy to verify that g preserves order and that $v = g \circ v_A$. □

2. Exponential and discrete valuations. Valuations in a field of rational functions.

The main goal of this section is to describe all valuations of a field of rational functions over a given field. We shall introduce the notion of exponential and discrete valuations and show that all interesting valuations in the rational functins field behave similarly to the well-known $p-$adic exponent. This requires some definitions. First, let G be an ordered abelian group. A subgroup H of G is said to be the **isolated subgroup** if:

$$\bigwedge_{h \in H} \{g \in G : 0 \leq g \leq h\} \subset H.$$

Clearly the trivial groups (the zero subgroup and the whole group G) are isolated. The set $\mathcal{G}(G)$ of all isolated subgroups of G is totally ordered by inclusion. The order type of the set $\mathcal{G}(G) \setminus \{G\}$ is called **the rank of** G. If G is a value group of some valuation v, then the **rank of valuation** v is the rank of G. Clearly v is a valuation of rank 0 if and only if G is the zero group. Moreover, $v : F \to G \cup \{\infty\}$ has rank less or equal that 1 if and only if G is Archimedean, that is:

$$\bigwedge_{a,b \in G} a, b \geq 0 \Rightarrow \bigvee_{n \in \mathbb{N}} (na \geq b).$$

Indeed, suppose that $G = \{0\}$. Then G is Archimedean and clearly the valuation v has rank 0. Suppose then that $G \neq \{0\}$. Fix $a\ inG$ and let $H_a = \bigcup_{n\in\mathbb{N}}\{b \in G : -na \leq b \leq na\}$. We shall show that H_a is an isolated subgroup of G.

Fix $h \in H_a$. Let $n \in \mathbb{N}$ be such number that $-na \leq h \leq na$. Let $g \in G$ be such that $0 \leq g \leq h$. Then $-na \leq 0 \leq g \leq h \leq na$ and therefore $g \in H_a$.

Now we shall show that if H an isolated subgroup of G and $a \in H$, then $H_a \subset H$. Indeed, fix an isolated subgroup H and let $a \in H$. Let $h \in H_a$ and $n \in \mathbb{N}$ be such number that $-na \leq h \leq na$. Obviously $na, -na \in H$. If $h > 0$, then since$0 \leq h \leq na$, we have that $h \in H$. If $h < 0$, then since $-na \leq h \leq 0$, we have that $0 \leq -h \leq na$, so $-h \in H$ and hence $h \in H$.

Thus H_a is the smallest isolated subgroup containing a. This implies that the rank of G is equal to 1 if and only if:

$$\bigwedge_{G\ni a>0} H_a = G,$$

which is equivalent to:

$$\bigwedge_{G\ni a>0}\ \bigwedge_{G\ni b>0}\ \bigvee_{n\in\mathbb{N}} b \leq na.$$

We can also define the **rank of a valuation ring.** If A is the valuation ring in F then the set:

$$\mathcal{B} = \{B : A \subset B,\ B \text{ - piercie waluacyjny}\}$$

is totally ordered by the inclusion relation. The order type of the set $\mathcal{B} \setminus \{F\}$ shall be called the rank of the ring A. We will show that the rank of valuation is equal to the rank of associated valuation ring. First we need to know some properties of isolated groups.

Theorem 4. *Let G and G_1 be ordered abelian groups, let H be an isolated subgroup of G.*

(1) *G/H is ordered by the relation:*

$$g + H \geq 0 + H \Leftrightarrow \bigvee_{h\in H} g \geq h.$$

The mapping $\kappa : G \to G/H$ given by:

$$\kappa(g) = g + H$$

is an order-preserving group homomorphism.

(2) *If $\phi : G \to G_1$ is an order-preserving group homomorphism, then $\ker\phi$ is an isolated subgroup of G.*

(3) *If $\phi : G \to G_1$ is an order-preserving group homomorphism, then $G/\ker\phi \cong \operatorname{im}\phi$.*

Proof. In order to prove (1) define $\kappa : G \to G/H$ by $\kappa(g) = g + H$ and let $S = \{a + G : a \geq 0\}$, $\overline{S} = \kappa(S)$. It is easy to verify that $\overline{S}$ is closed under addition. Observe that $\overline{S} \cap -\overline{S} = \{0\}$.

Indeed, fix $a_1 + H, a_2 + H \in \overline{S}$ and let $a_1 + H = -(a_2 + H)$. Thus $a_1, a_2 \in S$ and $\kappa(a_1) = -\kappa(a_2)$, hence $\kappa(a_1 + a_2) = 0 + H$, so $a_1 + a_2 \in H$. Next, since $a_1, a_2 \geq 0$, we have that $0 \leq a_1 \leq a_1 + a_2$ and since H is isolated, then $a_1 \in H$. It follows that $a_1 + H = \kappa(a_1) = 0 + H$.

In order to check that $\overline{S} \cup -\overline{S} = G/H$ fix $a + H \in G/H$. Then $a \in S$ or $-a \in S$, so $a + H \in \overline{S}$ or $-(a + H) \in \overline{S}$. Therefore $\overline{S}$ is an ordering in G/H and:

$$g + H \geq 0 + H \Leftrightarrow \kappa(g) \in \overline{S} \Leftrightarrow \bigvee_{a \in S} \kappa(g) = \kappa(a) \Leftrightarrow \bigvee_{a \in S} g - a \in H \Leftrightarrow \bigvee_{h \in H} g \geq h.$$

Obviously κ preservs ordering.

To prove (2) fix an arbitrary $h \in \ker \phi$ and let $g \in G$ be such that $0 \leq g \leq h$. Then $0 = \phi(0) \leq \phi(g) \leq \phi(h) = 0$, so $g \in H$. To finish the proof of the theorem observe that by the isomorphism theorem there exists an isomorphism $\psi : G/\ker \phi \to \operatorname{im} \phi$ such that $\kappa \circ \psi = \phi$. We shall show that ψ is order-preserving.

Fix $g \in \ker \phi$ and let $g + \ker \phi \geq 0 + \ker \phi$. Then there exists $h \in \ker \phi$ such that $g \geq h$. Therefore $\psi(g + \ker \phi) = \phi(g) \geq \phi(h) = 0$. □

Now we can prove that the rank of a valuation coincides with the rank of a valuation ring.

Theorem 5. *Let F be a field, G an ordered commutative group and $v : F \to G \cup \{\infty\}$ a Krull valuation. Let A_v be the valuation ring associated with v and let:*

$$\mathcal{B} = \{B : A_v \subset B, B \text{ - } ring\}.$$

1. *Every element of $\mathcal{B}$ is a valuation ring.*
2. *Let $v_B : F \to G_B \cup \{\infty\}$ be a Krull valuation associated with ring $B \in \mathcal{B}$ whose value group is G_B. Then there exists exactly one group homomorphism $g_B : G \to G_B$ such that $v_B = g_B \circ v$, g_B is an order-preserving surjection and* $\ker g_B = v(U(B))$.
3. *Let $\mathcal{G}(G)$ be the family of all isolated subgroups of G. Then the mapping $\Phi : \mathcal{B} \to \mathcal{G}(G)$ given by:*

$$\Phi(B) = \ker g_B$$

is an order-preserving bijection.

Proof. (1) is trivial: for $B \in \mathcal{B}$ fix $a \in F$ and suppose that $a \notin B$. Then also $a \notin A_v$, hence $a^{-1} \in A_v \subset B$.

To prove (2) fix $B \in \mathcal{B}$ and observe that $\mathsf{U}(A_v) \subset \mathsf{U}(B)$. The valuations v and v_B determine the group homomorphisms $v : \mathsf{U}(F) \to G$ and $v_B : \mathsf{U}(F) \to G$ such that $\mathsf{U}(A_v) = \ker v$, $\mathsf{U}(B) = \ker v_B$. By the isomorphism theorem there exists exactly one homomorphism $g_B : G \to G_B$ such that $v_B = g_B \circ v$. Moreover $\operatorname{im} g_B = \operatorname{im} v_B$ and $\ker g_B = v(\ker v_B) = v(\mathsf{U}(B))$. Since v_B is surjective it follows that also g_B is a surjection. It remains to show that g_B is order-preserving.

Fix $g \in G$ and let $g \geq 0$. Since v is a surjection, there exists $a \in F$ such that $g = v(a)$. Thus, since $v(a) = g \geq 0$, we have that $a \in A_v \subset B$. Suppose that $g_B(g) < 0$. Then $0 > g_B(g) = g_B(v(a)) = v_B(a)$, so $a \notin B$ - a contradiction.

In order to prove (3) observe that for all $B \in \mathcal{B}$ the group $\ker g_B$ is isolated, so Φ is well-defined. We shall pick an inverse function to Φ. Fix $H \in \mathcal{G}(G)$. By the previous theorem G/H is ordered. Define $\kappa_H : G \to G/H$ by $\kappa_H(g) = g + H$ and $w_H : F \to G/H \cup \{\infty\}$ by $w_H = \kappa_H \circ v$. It is easy to verify that w_H is a Krull valuation, so let B_H be a valuation ring corresponding to w_H. Since for $a \in A_v$ $w_H(a) = \kappa_H \circ v(a)$ and κ_H preserves an order, we get that $A_v \subset B_H$. Thus the mapping $\Psi : \mathcal{G}(G) \to \mathcal{B}$ given by:

$$\Psi(H) = B_H$$

is well-defined. We shall show that $\Psi \circ \Phi = \mathsf{id}_{\mathcal{B}}$.

Fix $B \in \mathcal{B}$, let $v_B : F \to G_B \cup \{\infty\}$ be the valuation associated with B, let $g_B : G \to G_B$ be such unique homomorphism that $v_B = g_B \circ v$, g_B is a surjection, preserves an order and $\ker g_B = v(\mathsf{U}(B))$. The mapping $\overline{g_B} : G/\ker g_B \to G_B$ given by $\overline{g_B}(g + \ker g_B) = g_B(g)$ is an isomorphism. Define the functions $\kappa_{\ker g_B} : G \to G/\ker g_B$ by $\kappa_{\ker g_B}(g) = g + \ker g_B$ and $w_{\ker g_B} : F \to G/\ker g_B \cup \{\infty\}$ by $w_{\ker g_B} = \kappa_{\ker g_B} \circ v$. Obviously $w_{\ker g_B}$ is a valuation. Since $v_B = g_B \circ v$, we have that $v_B = \overline{g_B} \circ w_{\ker g_B}$. By the theorem that describes the relationship between valuations and valuation rings, the valuation $w_{\ker g_B}$ corresponds to B.

Conversely, we shall show that $\Psi \circ \Phi = \mathsf{id}_{\mathcal{G}(G)}$. Fix $H \in \mathcal{G}(G)$ and let $B_H = \Psi(H)$ be such valuation ring corresponding to H that $A_v \subset B_H$. Let $v_{B_H} : F \to G_{B_H} \cup \{\infty\}$ be the valuation associated with the ring B_H. Define the mappings $\kappa_H : G \to G_H$ by $\kappa_H(g) = g + H$ and $w_H : F \to G/H \cup \{\infty\}$ by $w_H = \kappa_H \circ v$. Obviously w_H is a valuation. By the theorem describing the correspondence between valuations and valuation rings there exists an isomorphism $\tau : G/H \to G_{B_H}$ such that $v_{B_H} = \tau \circ w_H$. Thus $g : G \to G_{B_H}$ given by $g = \tau \circ \kappa_H$ is such homomorphism that $\ker g = H$ and $v_{B_H} = g \circ v$. Hence $g = g_{B_H}$ and $H = \ker g_{B_H}$. $\square$

Before we define discrete and exponential valuations, we need to know something about ordered Archimedean groups. We shall prove the following result:

Theorem 6 (Hölder)**.** *Every ordered Archimedean commutative group is isomorphic to some subgroup of the additive group of real numbers.*

Proof. Let G be such ordered abelian group that $\bigwedge_{g,h \in G} g, h \geq 0 \Rightarrow \bigvee_{n \in \mathbb{N}} (ng \geq h)$. Fix $g\ inG$ and let $g \geq 0$. Define the mapping $\Phi : G \to \mathbb{R}$ by:

$$\Phi(h) = \inf\{\frac{m}{n} \in \mathbb{Q} : mg > nh, m, n \in \mathbb{Z}\}.$$

Observe that Φ is well-defined. Indeed, the set $\bigwedge_{a,b \in G} a, b \geq 0 \Rightarrow \bigvee_{n \in \mathbb{N}} (na \geq b)$ is nonempty; because G is Archimedean, there exists $p \in \mathbb{Z}$ such that $pg > h$. Next, this set has a lower bound; since G is Archimedean, there is $q \in \mathbb{Z}$ such that $g < qh$. Now, if $mg > nh$, then $\frac{m}{n} \geq \frac{1}{q}$ - if $\frac{m}{n} < \frac{1}{q}$, then $mq < n$, so $mqg < ng < nqh$, hence $mg < nh$ - a contradiction.

We shall show that Φ is a homomorphism. Fix $h, k \in G$ and let $m, n, p, q \in \mathbb{Z}$ be such that $mg > nh$ and $pg > qk$. Since G is abelian $(mq + np)g = mqg + npg > nqh + qhk = nq(h + k)$. Thus $\frac{m}{n} + \frac{p}{q} = \frac{mq+np}{nq} \geq \Phi(h + k)$. Suppose that $\Phi(h + k) < \Phi(h) + \Phi(k)$. Then we may pick numbers $m, n, p, q \in \mathbb{Z}$ such that $\frac{m}{n} < \Phi(h)$, $\frac{p}{q} < \Phi(k)$ and $\Phi(h + k) < \frac{m}{n} + \frac{p}{q} < \Phi(h) + \Phi(k)$. Hence $mg \leq nh$ and $pg \leq qk$ and since G is commutative, this yields $(mq + np)g = mqg + npq \leq nqh + nqk = nq(h + k)$. Thus $\frac{m}{n} + \frac{p}{q} = \frac{mq+np}{nq} < \Phi(h + k)$ - which is a contradiction.

It is obvious that Φ is order-preserving, so it remains to show that Φ is one-to-one. Fix $h \in G$. If $h > 0$, then there exists $n \in \mathbb{N}$ such that $nh > g$, so $\Phi(h) \leq \frac{1}{n} > 0$. If $h < 0$, then there exists $n \in \mathbb{N}$ such that $-nh > g$, so $-g > nh$, and thus $\Phi(h) \leq -\frac{1}{n} < 0$. Therefore $\ker \Phi = \{0\}$. $\square$

Now we can introduce the notion of **exponential valuations.** The valuation whose value group is a subgroup of the additive group of real numbers is called the exponential valuation. Observe that - according to the above theorems - for a given valuation $v : F \to G \cup \{\infty\}$ the following four conditions are equivalent:

(1) v has rank less or equal than 1,

(2) the valuation ring A_v has rank less or equal that 1,
(3) G is Archimedean,
(4) v is an exponential valuation.

An exponential valuation whose value group is a discrete subspace of $\mathbb{R}$ (with respect to the usual topology in $\mathbb{R}$) is called the **discrete valuation**. We need some more information about discrete subspaces of $\mathbb{R}$. Let G be a subgroup of $\mathbb{R}$. We shall prove that the following conditions are equivalent:

(1) G is a discrete subspace of $\mathbb{R}$,
(2) G is not dense in $\mathbb{R}$,
(3) $\{g \in G : g > 0\}$ has a minimal element,
(4) $G = \rho \cdot \mathbb{Z}$ for some $\rho > 0$.

(1) $\Rightarrow$ (2): Suppose that G is a discrete subspace of $\mathbb{R}$. Then $\{\{a\} : a \in G\}$ is a basis of the topology in G and $\{(b, c) : b, c \in \mathbb{Q}, b < c\}$ is a basis of the topology in $\mathbb{R}$. On the other hand the topology in G is induced from $\mathbb{R}$, so:

$$\bigwedge_{a \in G} \bigvee_{b,c \in \mathbb{Q}} \{a\} = (b, c).$$

So if $a \in G$, then we may pick $b \in \mathbb{Q}$ such that $(b, a) \cap G = \emptyset$. Thus G cannot be dense in $\mathbb{R}$.

(2) $\Rightarrow$ (3): suppose that $\{g \in G : g > 0\}$ has no minimal element. We shall show that G is dense in $\mathbb{R}$. Fix $(a, b) \subset \mathbb{R}$. We may assume that $a, b > 0$. Since $\{g \in G : g > 0\}$ has no minimal element, we may choose $g \in G$ such that $0 < g < \frac{b-a}{2}$. Since in the group $\mathbb{R}$ the Archimedean rule holds, there exists $n \in \mathbb{Z}$ such that $ng < b \leq (n + 1)g$.Observe that $ng > a$ - otherwise, $(n + 1)g < a + g < b$, a contradiction.

(3) $\Rightarrow$ (4): let g_0 be the minimal element of the set $\{g \in G : g > 0\}$. Fix $g \in G$. By the Archimedean rule applied for $\mathbb{R}$, there exists $n \in \mathbb{Z}$ such that $ng_0 \leq g < (n + 1)g_0$, so $0 \leq g - ng_0 < g_0$. By the choice of g_0, $g = ng_0$, so $G = g_0 \cdot \mathbb{Z}$.

(4) $\Rightarrow$ (1): observe that for all $a \in G (a - \rho, a + \rho) \cap G = \{a\}$. Therefore $\{\{a\} : a \in G\}$ is a basis for the topology of G.

We will try to simplify the notion of discrete valuations. A discrete valuation is said to be the **normalized discrete valuation** when its value group is $\mathbb{Z}$. Let F be a field, G_1 and G_2 ordered abelian subgroups of $\mathbb{R}$ and $v_1 : F \to G_1 \cup \{\infty\}$, $v_2 : F \to G_2 \cup \{\infty\}$ valuations. Clearly, if there exists a real number $b \in \mathbb{R}$ such that $v_1(a) = b \cdot v_2(a)$ for all $a \in F$, then v_1 and v_2 are equivalent. Actually, the converse is also true; let $\phi : G_1 \to G_2$ be such group isomorphism that $v_1 = \phi \circ v_2$. By the previous remark, $G_1 = \rho_1 \cdot \mathbb{Z}$, $G_2 = \rho_2 \cdot \mathbb{Z}$. We shall show that $\phi(\rho_1) = \rho_2$.

Suppose that $\phi(\rho_1) = n\rho_2$ for some $n \in \mathbb{N} \setminus \{1\}$. Since ϕ is an isomorphism, there exists $m \in \mathbb{N}$ such that $\phi(m\rho_1) = (n - 1)\rho_2$. But $\phi(m\rho_1) = \phi(\rho_1) + \ldots + \phi(\rho_1) = mn\rho_2$. Hence $mn = n1$, that is $n(1 - m) = 1$, so $n = 1$ and $m = 0$, a contradiction.

Therefore $\phi(m\rho_1) = m\rho_2 = \frac{\rho_2}{\rho_1} m\rho_1$ and taking $b = \frac{\rho_2}{\rho_1}$ we obtain $v_1(a) = \phi(v_2(a)) = \frac{\rho_2}{\rho_1} v_2(a)$.

This remark shows that every discrete valuation is equivalent to exactly one normalized discrete valuation. Now we are going to describe the set of all normalized discrete valuations in the field of rational functions. First we state a bit more general result.

Theorem 7. *Let R be a unique factorization domain, let F be the field of fractions of R, let $\mathbb{P}$ be the set of representatives of irreducible elements in R (that is every irreducible element in R is associated with exactly one $P \in \mathbb{P}$).*

(1) *For every $P \in \mathbb{P}$ the mapping $v_P : F \to \mathbb{R} \cup \{\infty\}$ given by:*

$$v_P(a) = \begin{cases} \infty & \text{if } a = 0, \\ n_P & \text{if } a = u \prod_{Q \in \mathbb{P}} Q^{n_Q}, n_Q \in \mathbb{Z}, u \in \mathsf{U}(R), \end{cases}$$

is a normalized discrete valuation in F. Moreover:

$$A_{v_P} = R_{(P)} \supset R,\ \ M_{v_P} = (P) \text{ w } A_{v_P},\ \ (P) = M_{v_P} \cap R \text{ w } R,\ \ R/(P) = \kappa_{v_P}(R) \subset F_{v_P},$$

where $\kappa_{v_P} : A_{v_P} \to F_{v_P}$ is a cannonical epimorphism given by:

$$\kappa_{v_P}(a) = a + M_{v_P}.$$

(2) $R = \bigcap_{P \in \mathbb{P}} A_{v_P}$.

(3) *For all $a \in F$ the set:*

$$\{P \in \mathbb{P} : v_P(a) \neq 0\}$$

is finite.

(4) *If R is a principle ideal domain, then for all $P \in \mathbb{P}$ $\kappa_{v_P}(R) = F_{v_P}$. Moreover, if $v : F \to \mathbb{R} \cup \{\infty\}$ is a non-trivial exponential valuation such that $R \subset A_v$, then v is equivalent to exactly one valuation of the form v_P for some $P \in \mathbb{P}$.*

Proof. First we shall prove (1). It is trivial to check that v_P is a valuation. We shall show that $R \subset A_{v_P}$. Indeed, fix $a \in R$. Then $a = u \prod_{Q \in \mathbb{P}} Q^{n_Q}$, $n_Q \in \mathbb{Z}$, $u \in \mathsf{U}(R)$ or $a = 0$. Thus $v_P(a) = n_P \geq 0$ or $v_P(a) = \infty > 0$, that is $a \in A_{v_P}$. Next, we shall prove that $A_{v_P} = R_{(P)}$. Indeed:

$$R_{(P)} = \{\frac{s}{t} : s, t \in R, t \notin (P)\} = \{\frac{s}{t} : s, t \in R, P \nmid t\} = \{\frac{s}{t} : s, t \in R, v_P(\frac{s}{t}) \geq 0\} = A_{v_P}.$$

In order to prove that in the ring A_{v_P} the identity $M_{v_P} = (P)$ holds, let us observe that:

$$M_{v_P} = \{\frac{s}{t} : s, t \in R, v_P(\frac{s}{t}) > 0\} = \{\frac{s}{t} : P \nmid t, P | s\} = (P).$$

To prove that $(P) = M_{v_P} \cap R$ holds in the ring R note that:

$$(P) = \{aP : a \in R\} = \{\frac{a}{1} \cdot P : a \in R\} = M_{v_P} \cap R.$$

Finally, the identity $R/(P) = \kappa_{v_P}(R)$ is true, since:

$$R/(P) = \{a + (P) : a \in R\} = \{a + M_{v_P} : a \in R\} = \kappa_{v_P}(R).$$

The inclusion $\kappa_{v_P}(R) \subset F_{v_P}$ is obvious. (2) is true since $R \subset A_{v_P}$ for all $P \in \mathbb{P}$, and (3) follows directly from the definiton of a unique factorization ring and a field of fractions. It remains to show (4).

Suppose that R is a principal ideal domain. First we shall show that $\kappa_{v_P}(R) = F_{v_P}$. To do so, let us fix $a + M_{v_P} \in F_{v_P}$. There exists an element $\frac{s}{t} \in A_{v_P} = R_{(P)}$, $s, t \in R$, $t \notin (P)$ such that $\kappa_{v_P} = a + M_{v_P}$. Since R is a principal ideal domain and $t \notin (P)$, we have that $NWD(P, t) = 1$, so there exist $c, d \in R$ such that $cP + td = 1$. Thus $\frac{s}{t} - sd = \frac{s}{t}(1 - dt) = \frac{s}{t}(cp + td - td) = \frac{s}{t}cp \in (P) = M_{v_P}$. Hence $\kappa_{v_P}(\frac{s}{t} - sd) = \kappa_{v_P}(\frac{s}{t}) - \kappa_{v_P}(sd) = M_{v_P}$, that is $\kappa_{v_P}(\frac{s}{t}) = \kappa_{v_P}(sd) = a + M_{v_P}$ and $sd \in R$.

Let $v : f \to \mathbb{R} \cup \{\infty\}$ be a non-trivial exponential valuation such that $R \subset A_v$. Then M_v is the maximal ideal in the ring A_v, so it is a prime ideal. Hence $M_v \cap R$ is a prime ideal in the ring R, so $M_v \cap R = (P)$ for some $P \in \mathbb{P}$. Since R is a principal ideal domain, $M_v \cap R$ is also maximal and thus non-zero. Let $v(P) = \rho$. Since v is non-trivial, we have that $P \notin \mathsf{U}(A_v)$ and hence $P \notin \mathsf{U}(R)$, which implies that $\rho > 0$. Thus $v(P) = \rho = \rho v_P(P)$. Moreover, for $a \in R \setminus (P)$ $v(a) = 0 = \rho v_P(a)$. Since any non-zero element $x \in F$ is of the form $x = P^m \frac{a}{b}$, $m \in \mathbb{Z}$, $a, b \in R \setminus (P)$, we have:

$$v(x) = v(P^m) + v(a) - v(b) = mv(P) = m\rho = \rho v_P(x),$$

which means that v is equivalent to v_P. Since v_P and v_Q for $P \neq Q$, $P, Q \in \mathbb{P}$ are not equivalent to each other, the choice of $P \in \mathbb{P}$ is unique. $\square$

As a corollary we shall state the following theorem describing normalized exponential valuations of a field of rational functions.

Theorem 8. *Let F be a field, z a transcendental element over F and let us consider the field $F(z)$. Let $\mathbb{P}$ be the set of all irreducible polynomials in the ring $F[X]$, let $R = F[z]$. For an arbitrary $P \in \mathbb{P}$ define the mapping $v_{z,P} : F(z) \to \mathbb{R} \cup \{\infty\}$ by:*

$$v_{z,P}(a) = \begin{cases} \infty & \text{if } a = 0, \\ n_P & \text{if } a = u \prod_{Q \in \mathbb{P}} Q(z)^{n_Q}, n_Q \in \mathbb{Z}, u \in \mathsf{U}(R). \end{cases}$$

Define also the function $v_{z,\infty} : F(z) \to \mathbb{R} \cup \{\infty\}$ by:

$$v_{z,\infty}(a) = \begin{cases} \infty, & \text{if } a = 0, \\ \deg g - \deg f, & \text{if } a = \frac{f(z)}{g(z)}, f, g \in F[X] \setminus \{0\}. \end{cases}$$

We set $\deg \infty = 1$.

(1) *The mapping*
$$P \mapsto v_{z,P}$$
establishes a bijection between the set $\mathbb{P} \cup \{\infty\}$ and the set of all normalized exponential valuations of the field $F(z)$ such that $v(a) = 0$ for $a \in F$. In particular, every non-trivial exponential valuation in $F(z)$ is discrete.

(2) $\bigcap_{P \in \mathbb{P} \cup \{\infty\}} A_{v_{z,P}} = F$.

(3) *For every $P \in \mathbb{P} \cup \{\infty\}$ the residue field $F_{v_{z,P}}$ of the valuation $v_{z,P}$ is a simple extension of the field F (more precisely - a simple extension of an isomorphic image of the field F, $\kappa_{v_{z,P}}(F)$, where $\kappa_{v_{z,P}} : A_{v_{z,P}} \to F_{v_{z,P}}$ is the cannonical epimorphism). Moreover $[F_{v_{z,P}} : \kappa_{v_{z,P}}(F)] = \deg P$.*

Proof. In order to prove (1) we first observe the trivial fact that $v_{z,\infty}$ is a discrete valuation. By the previous results it suffices to show that if $v : F(z) \to \mathbb{R} \cup \{\infty\}$ is such exponential valuation that $R \not\subseteq A_v$, then v is equivalent to the valuation $v_{z,\infty}$. Since $R = F[z] \not\subseteq A_v$ and $F \subset A_v$ (because $v(a) = 0$ for $a \in F$), we have that $z \notin A_v$. Thus $z^{-1} \in A_v$. Let $v(z^{-1}) = \rho$. Obviously $\rho > 0$ - if $\rho = 0$, then $v(z) = v(z^{-1}) = 0$ and $z \in A_v$. In particular $v(z) = -\rho$. Let $f = a_n X^n + \ldots + a_0 \in F[X]$. Then, since $v(a_k z^k) = -\rho k \neq \rho l = v(a_l z^l)$ for $k \neq l$, we get:

$$v(f(z)) = v(a_n z^n + \ldots + z_0) = \min\{v(a_n z^n), \ldots, v(a_0)\} = \min\{-n\rho, \ldots, -\rho, 0\} = -n\rho.$$

Thus, when $x = \frac{f(z)}{g(z)} \in F(z)$, we obtain:

$$v(x) = v(f(z)) - v(g(z)) = -\deg f\rho + \deg g\rho = \rho v_{z,\infty}(x).$$

It is clear that $\bigcap_{P\in\mathbb{P}\cup\{\infty\}} A_{v_{z,P}} = \bigcap_{P\in\mathbb{P}} A_{v_{z,P}} \cap A_{v_{z,\infty}} = F[z] \cap A_{v_{z,\infty}} = F$, which proves (2), so it remains to show (3). Fix $P \in \mathbb{P}$ and observe that $\kappa_{v_{z,P}} \restriction_F \colon A_{v_{z,P}} \to F_{v_{z,P}}$ is - as a non-trivial field homomorphism - an embedding, so $F \cong \kappa_{v_{z,P}}(F)$. Since $F[z]$ is a principle ideal domain, we have that $F_{v_{z,P}} = \kappa_{v_{z,P}}(F[z]) = \kappa_{v_{z,P}}(F)[\kappa_{v_{z,P}}(z)]$. Obviously if $P = a_nX^n + \ldots + a_0$, then $\kappa_{v_{z,P}}(a_n)X^n + \ldots + \kappa_{v_{z,P}}(a_0)$ is a minimal polynomial for $\kappa_{v_{z,P}}(z)$ and $[F_{v_{z,P}} : \kappa_{v_{z,P}}(F)] = n = \deg P$.

When $P = \infty$, then:

$$v_{z,\infty}(\frac{a_nz^n + \ldots + a_0}{b_mz^m + \ldots + b_0}) = m-n = v_{z^{-1},X}(z^n)-v_{z^{-1},X}(z^m) = v_{z^{-1},X}(\frac{a_nz^n + \ldots + a_0}{b_mz^m + \ldots + b + 0})$$

and the result follows from the previous part of proof and the remark that $\deg \infty = \deg X = 1$. $\square$

Before we go to further theorems, we shall illustrate the developed theory with some examples.

(1) Let $F = \mathbb{C}$. Then the set $\mathbb{P}$ of irreducible polynomials in $\mathbb{C}[X]$ is just the set of linear polynomials of the form:

$$\{X - a : a \in \mathbb{C}\}.$$

The construction of valuation is clear, the residue fields associated with valuations derived from polynomials $X - a$ are just the complex numbers, since $[\mathbb{C}_{v_{z,X-a}} : \kappa_{v_{z,X-a}}(\mathbb{C})] = \deg(X - a) = 1$. Similarly the residue field of the valuation $v_{z,\infty}$ is $\mathbb{C}$.

(2) Let $F = \mathbb{R}$. Then the set $\mathbb{P}$ of irreducible polynomials in $\mathbb{R}[X]$ consists of the polynomials of the form:

$$X - a \text{ for some } a \in \mathbb{R} \quad or \quad (X - a)^2 + b^2 \text{ for some } a, b \in \mathbb{R}.$$

Indeed, since $\mathbb{C} \supset \mathbb{R}$ is an extension of degree 2 and $\mathbb{C}$ is algebraically closed, every polynomial decomposes into irreducible factors of degree 1 or 2. If $X^2 - 2aX + c$ is an irreducible polynomial, then $4a^2 - 4c < 0$, so $c - a^2 > 0$. Moreover $X^2 - 2aX + c = (X - a)^2 + (c - a)^2$, thus taking $c - a^2 = b^2$ we get the polynomial of the form $(X - a)^2 + b^2$.

Next, we have that:

$$[\mathbb{R}_{v_{z,X-a}} : \kappa_{v_{z,X-a}}(\mathbb{R})] = \deg(X - a) = 1$$

so the residue field $\mathbb{R}_{v_{z,X-a}}$ is just $\mathbb{R}$. Similarly $\mathbb{R}_{v_{z,\infty}} = \mathbb{R}$ and since:

$$[\mathbb{R}_{v_{z,(X-a)^2+b^2}} : \kappa_{v_{z,(X-a)^2+b^2}}(\mathbb{R})] = \deg((X - a)^2 + b^2) = 2$$

we have $\mathbb{R}_{v_{z,(X-a)^2+b^2}} = \mathbb{C}$. Since the polynomials $X - a$ and the element ∞ corresponds with points $a \in \mathbb{R}$ and the point "at infinity", we shall often call the unique maximal ideals associated with valuation rings derived from valuations related with such polynomials to be the **real places** of the field $\mathbb{R}(X)$.

(3) Let $F = \mathbb{Q}$. Then the set $\mathbb{P}$ of irreducible polynomials may contain polynomials of any finite degree, so the residue fields associated with valuations derived from the polynomials in $\mathbb{P}$ are the finite extensions of the field $\mathbb{Q}$, that is - the algebraic number fields. By the primitive element theorem, such fields are simply generated, so we may associate with each valuation in $\mathbb{Q}(X)$ an algebraic number $\alpha \in \mathbb{C}$.

Now we want to show that all "interesting" valuations in a field of rational functions can be described in the above manner. In order to do that we need the so called Krull's intersection theorem. The nice and short proof given here is taken from [3].

Theorem 9 (Krull)**.** *Let R be a noetherian ring and I an ideal in R. Then $\bigcap_{n=1}^{\infty} I^n = (0)$ if and only if no element of the set $\{1 - a : a \in I\}$ is a zero-divisor.*

Proof. ($\Rightarrow$) Suppose that $1 - z$, $z \in I$ is a zero divisor. Then $(1 - z)y = 0$ for some $y \neq 0$. This implies that $y = zy = z^2 y = \ldots = z^n y$, $n \in \mathbb{N}$, that is $y \in \bigcap_{n=1}^{\infty}$.

($\Leftarrow$) Since R is noetherian, the ideal I is finitely generated and we may take $I = (a_1, \ldots, a_k)$ for some $a_1, \ldots, a_k \in R$. Since $b \in \bigcap_{n=1}^{\infty} I^n$, for every $n \in \mathbb{N}$ there is a homogeneous polynomial $P_n(X_1, \ldots, X_K)$ of degree n such that:

$$b = P_n(a_1, \ldots, a_k).$$

Now define the ideals $J_n = (P_1, \ldots, P_n)$. Clearly the family $\{J_n : n \in \mathbb{N}\}$ forms an ascending chain of ideals in the ring $R[X_1, \ldots, X_k]$. By the Hilbert basis theorem, the ring $R[X_1, \ldots, X_k]$ is also noetherian, so there exists a number $m \in \mathbb{N}$ such that $J_m = J_{m+1}$. That means, that:

$$P_{m+1} = Q_m P_1 + \ldots + Q_1 P_m$$

where Q_i are homogeneous polynomials of degree i. Substituting $X_1 = a_1, \ldots, X_k = a_k$ gives:

$$b = b(Q_1(a_1, \ldots, a_k) + \ldots |Q_m(a_1, \ldots, a_k))$$

or equivalently

$$b \cdot [1 - (Q_1(a_1, \ldots, a_k) + \ldots |Q_m(a_1, \ldots, a_k))] = 0.$$

Q_i are homogeneous of positive degree, so $Q_i(a_1, \ldots, a_k) \in I$. Since I is a proper ideal, $1 \notin I$, which proves that $1 - (Q_1(a_1, \ldots, a_k) + \ldots |Q_m(a_1, \ldots, a_k)) \neq 0$. By our assumption such element is not a zero divisor, so $b = 0$. □

As a corollary observe that if R is a noetherian domain and I an ideal in R, then $\bigcap_{n=1}^{\infty} = (0)$. Now we shall prove the following result, which characterizes discrete valuation rings.

Theorem 10. *Let R be a local domain, let K be its field of fractions. Then the following are equivalent:*

(1) *R is a discrete valuation ring in K,*
(2) *R is a noetherian valuation ring in K,*
(3) *R is a principal ideal domain,*
(4) *R is a noetherian ring and its only maximal ideal is a principal ideal,*
(5) *the only maximal ideal I of R is principal and $\bigcap_{n=1}^{\infty} I^n = (0)$.*

Proof. (1) $\Rightarrow$ (2): Suppose that R is a local discrete valuation domain. We may assume that R is a normalized valuation ring. We shall show that R is a principal ideal domain (so it is noetherian). Let $v : K \to \mathbb{Z}$ be such valuation in a field K, that $R = \{a \in K : v(a) \geq 0\}$. Let $p \in K$ be such element that $v(p) = 1$. If $r \in R$, then $v(r) = k = kv(p) = v(p^k)$ for some $k \in \mathbb{N}$. Thus $v(r) - v(p^k) = v(\frac{r}{p^k}) = 0$

and hence $\frac{r}{p^k} \in \mathsf{U}(R)$, so $r = up^k$ for some $u \in \mathsf{U}(R)$. Therefore $(r) = (p^k)$. Fix an ideal $I \lhd R$, $\{0\} \subsetneq I \subsetneq R$. Then:

$$I = \bigcup_{r \in I \setminus \{0\}} (r) = \bigcup_{r \in I \setminus \{0\}} (p^{v(r)}).$$

Obviously $\mathcal{L} = \{(p^{v(r)}) : r \in I \setminus \{0\}\}$ is a chain of ideals and if $v(r_1) \leq v(r_2)$, then $p^{v(r_1)} | p^{v(r_2)}$ and then $(p^{v(r_1)}) \supset (p^{v(r_2)})$. Since I is a proper ideal, such chain has an upper bound. Therefore there exists $r_0 \in I \setminus \{0\}$ such that $v(r_0) = \min\{v(r) : r \in I \setminus \{0\}\}$. This implies:

$$I = \bigcup_{r \in I \setminus \{0\}} (p^{v(r)}) = (p^{v(r_0)}).$$

(2) $\Rightarrow$ (3): Suppose that R is a noetherian valuation ring. We shall show that R is a principal ideal domain. Let I be an ideal in the ring R. Then $I = (a_1, \ldots, a_m)$. Let $v : K \to G$, be such that $R = \{a \in K : v(a \geq 0)\}$, where G is some ordered abelian group. In the set $\{v(a_1), \ldots, v(a_m)\}$ there exists the least element, say $v(a_1)$. Then $v(a_i) - v(a_1) = v(\frac{a_i}{a_1}) \geq 0$ which implies $\frac{a_i}{a_1} \in R$, that is $a_i = u_i a_1$ for some $u_i \in R$, $i \in \{2, \ldots, m\}$. That means $a_1, \ldots, a_m \in (a_1)$, so $I \subset (a_1)$. Obviously the second inclusion is always true, so $I = (a_1)$.

(3) $\Rightarrow$ (4) is clear and (4) $\Rightarrow$ (5) is just a corollary from the Krull's intersection theorem.

(4) $\Rightarrow$ (5): Suppose that $I = (p)$ is the only maximal ideal in the ring R and that $\bigcap_{n=0}^{\infty} I^n = (0)$. We shall show that R is a discrete valuation ring in the field K. Fix $x \in R \setminus \{0\}$. Since $\bigcap_{n=0}^{\infty} I^n = (0)$, the set $\{n \in \mathbb{N} : x \in I^n\} = \{n \in \mathbb{N} : x \in (p^n)\}$ has a maximum. Define the function $v : R \to \mathbb{N} \cup \{\infty\}$ by:

$$v(x) = \begin{cases} \infty, & \text{if } x = 0, \\ \max\{n \in \mathbb{N} : x \in (p^n)\}, & \text{if } x \neq 0. \end{cases}$$

We shall show that $v(x + y) \geq \min\{v(x), v(y)\}$, $x, y \in R$. If $x = 0$ or $y = 0$ or $x = y = 0$, this is obvious. Suppose that $x \neq 0$ and $y \neq 0$. Let k, l, m be the smallest numbers such that $x + y \in (p^k)$, $x \in (p^l)$, $y \in (p^m)$. We may assume that $l \leq m$. Then $(p^l) \supset (p^m)$, so $x, y \in (p^l)$ and hence $x + y \in (p^l)$. Thus $l \leq k$ and so $\min\{l, m\} \leq k$.

We shall show that $v(xy) = v(x) + v(y)$. If $x = 0$ or $y = 0$ or $x = y = 0$, this is obvious. Suppose that $x \neq 0$ and $y \neq 0$. Since $x \in (p^{v(x)})$ i $y \in (p^{v(y)})$, we have that $x = u_1 p^{v(x)}$ and $y = u_2 p^{v(y)}$, $u_1, u_2 \in R$. Thus $xy = u_1 u_2 p^{v(x)+v(y)}$, so $xy \in (p^{v(x)+v(y)})$ and hence $v(xy) \geq v(x) + v(y)$. If $v(xy) > v(x) + v(y)$, then $xy \in (p^{v(x)+v(y)+1})$, so $x + y = u_3 p p^{v(x)+v(y)}$. On the other hand $xy = u_1 u_2 p^{v(x)+v(y)}$, so $u_1 u_2 p^{v(x)+v(y)} = u_3 p p^{v(x)+v(y)}$, thus $p^{v(x)+v(y)}(u_1 u_2 - u_3 p) = 0$ and since R is a domain and $p \neq 0$, this implies that $u_1 u_2 - u_3 p = 0$, so $u_1 u_2 = u_3 p$, that is $u_1 u_2 \in (p)$. Since (p) is maximal, it is prime. Thus $u_1 \in (p)$ or $u_2 \in (p)$ - we may assume that $u_1 \in (p)$. Then $u_1 = u_4 p$ for some $u_4 \in R$, and hence $x = u_1 p^{v(x)} = u_4 p^{v(x)+1}$, so $x \in (p^{v(x)+1})$ - which contradicts the definition of v.

Define the mapping $\tilde{v} : K \to \mathbb{Z} \cup \{\infty\}$ by:

$$\tilde{v}(\frac{a}{b}) = v(a) - v(b).$$

It is easy to check that $\tilde{v}$ is a discrete valuation. Clearly $R \subset A_{\tilde{v}}$, so it remains to show the other inclusion. Fix $\frac{x}{y} \in A_{\tilde{v}}$ and let $x = u_1 p^{v(x)}$, $y = u_2 p^{v(y)}$ for some

$u_1, u_2 \in R$. If $\tilde{v}(\frac{x}{y}) = n \geq 0$, then $v(x) - v(y) = n$, so $v(x) = v(y) + n$. Thus $x = u_1 p^{v(x)} = u_1 p^{v(y)+n}$, hence $xu_2 = u_1 u_2 p^{v(y)} p^n = u_1 y p^n$, that is $\frac{x}{y} = \frac{u_1}{u_2} p^n$. Moreover, observe that $\frac{u_1}{u_2} \in R$. Indeed, suppose that $\frac{u_1}{u_2} \notin R$. Since $u_1 \in R$, we get $u_2^{-1} \notin R$. But $u_2 \in R$, so $u_2 \notin \mathsf{U}(R)$. That means that $u_2 \in I = (p)$, that is $u_2 = u_3 p$ for some $u_3 \in R$. Then $y = u_2 p^{v(y)} = u_3 p^{v(y)+1}$, which contradicts the definition of v. Thus $\frac{u_1}{u_2} \in R$, $p^n \in R$ which gives $\frac{x}{y} \in R$. □

Next we shall define the class of algebraic function fields and show that all valuation rings in such fields are characterized in the way described before. The **algebraic function field** in one variable over the field K is the field F such that $F \supset K$ and $F \supset K(x)$ is a finite extension for some element $x \in F$ transcendental over K. The algebraic closure $\hat{K}$ of the field K in F shall be called the **field of constans.** Before we prove our main result we need some lemmas.

Lemma 1. *Let F be an algebraic function field over K. Then $z \in F$ is transcendental over K if and only if $[F : K(z)] < \infty$.*

Proof. ($\Leftarrow$): Suppose that $[F : K(z)] < \infty$ and z is algebraic over K. Then $[F : K] = [F : K(z)][K(z) : K] < \infty$, so the extension $F \supset K$ is finite - a contradiction.

($\Rightarrow$): Suppose that $F \supset K(x)$ is finite and x and z are transcendental over K. Consider the extensions $K(x) \subset K(z,x) \subset F$. Since $K(x) \subset F$ is a finite extension, we have that $K(x) \subset K(z,x)$ and $K(z,x) \subset F$ are also finite. Thus z is algebraic over $K(x)$. We shall show that x is algebraic over $K(z)$.

Since z is algebraic over $K(x)$, there exists $f \in K(x)[X]$ such that $f(z) = 0$. Since $K(x) = (K[x])$, we may suppose that $f \in K[x][X]$. Let $F(t,X) \in K[t,X]$ be such polynomial that $F(x,X) = f(X)$. Since x is transcendental over K, we have that F is non-zero. Let $g(t) = F(t,z) \in K(z)[t]$. Since z is transcendental over K, g is non-zero. Moreover, $g(x) = F(x,z) = f(z) = 0$. So x is algebraic over $K(z)$.

Consider the extensions $K(z) \subset K(x,z) \subset F$. Since $K(z) \subset K(z,x)$ is algebraic and finitely generated, it is finite. Since $K(z,x) \subset F$ is finite, $K(z) \subset F$ is also finite. □

Lemma 2. *Let F be an algebraic function field over K, let R be such valuation ring in F that $K \subsetneq R \subsetneq F$. Then:*

$$\hat{K} \subset R \quad \text{and} \quad \hat{K} \cap (R \setminus \mathsf{U}(R)) = \{0\}.$$

Proof. Fix $z \in \hat{K}$ and suppose that $z \notin R$. Then $z^{-1} \in R$. Since z is algebraic over K, we have that z^{-1} is algebraic over K, so for some $a_1, \ldots, a_r \in K$:

$$a_r(z^{-1})^r + \ldots + a_1 z^{-1} + 1 = 0,$$

hence $-1 = z^{-1}(a_r(z^{-1})^{r-1} + \ldots + a_1)$. Thus $z = -(a_r(z^{-1})^{r-1} + \ldots + a_1) \in K(z^{-1}) \subset R$, so $z \in R$ - a contradiction.

Suppose that there exists $a \neq 0$ such that $a \in \hat{K} \cap (R \setminus \mathsf{U}(R))$. Since $\hat{K}$ is a field, we also have that $a^{-1} \in \hat{K} \subset R$. Since $R \setminus \mathsf{U}(R)$ is an ideal in R, we have that $aa^{-1} \in R \setminus \mathsf{U}(R)$, so $1 \in R \setminus \mathsf{U}(R)$, which is a contradiction. □

Lemma 3. *Let F be an algebraic function field over K, let R be such valuations ring in F that, $K \subsetneq R \subsetneq F$. Let $0 \neq x \in R \setminus \mathsf{U}(R)$ and let $x_1, \ldots, x_n \in R \setminus \mathsf{U}(R)$ be such elements that $x_1 = x$ and $x_i \in x_{i+1} R \setminus \mathsf{U}(R)$, $i \in \{1, \ldots, n-1\}$. Then $n \leq [F : K(x)] < \infty$.*

Proof. By the previous lemma x is not algebraic over K, so it is transcendental. By the first lemma $[F : K(x)] < \infty$. It remains to show that $x_1, \ldots, x_n$ are linearly independent over $K(x)$. Suppose that there exist $f_1, \ldots, f_n \in K(x)$ not all zero such that:

$$f_1 x_1 + \ldots + f_n x_n = 0.$$

The elements f_i are rational functions in one indeterminate x. Multiplying both sides of the above equality by the common denominator of $f_1, \ldots, f_n$ we may assume that $f_1, \ldots, f_n \in K[x]$. Eventually dividing by the appropriate power of x we may also assume that not all f_i are divisibe by x. Let $a_i = f_i(0)$, $i \in \{1, \ldots, n\}$, be the free coefficients of the polynomials $f_1, \ldots, f_n$. Let a_j be the last non-zero element in the sequence $a_1, \ldots, a_n$. Then:

$$-f_j x_j = f_1 x_1 + \ldots + f_{j-1} x_{j-1} + f_{j+1} x_{j+1} + \ldots + f_n x_n.$$

Moreover, by the choice of j we get $f_i = x g_i$ for some $g_i \in K[x]$, $i \in \{j+1, \ldots, n\}$. Dividing both sides of the above equality by x_j yields:

$$-f_j = f_1 \frac{x_1}{x_j} + \ldots + f_{j-1} \frac{x_{j-1}}{x_j} + x g_{j+1} \frac{x_{j+1}}{x_j} + \ldots + x g_n \frac{x_n}{x_j}.$$

Since $x = x_1 \in R \setminus \mathsf{U}(R) \subset R$ and $K \subset R$, we get $f_1, \ldots, f_n \in K[x] \subset R$. Since $x_i \in x_{i+1} R \setminus \mathsf{U}(R)$, $i \in \{1, \ldots, n-1\}$, we have in particular $x_i \in x_j R \setminus \mathsf{U}(R)$ for $i \in \{1, \ldots, j-1\}$. Hence $\frac{x_i}{x_j} \in R \setminus \mathsf{U}(R)$ and so $f_i \frac{x_i}{x_j} \in R \setminus \mathsf{U}(R)$ for $i \in \{1, \ldots, j-1\}$. Similarly $\frac{x}{x_j} = \frac{x_1}{x_j} \in R \setminus \mathsf{U}(R)$ and since $x_i, g_i \in R$ for $i \in \{j+1, \ldots, n\}$, we have that $\frac{x}{x_j} g_i x_i \in R \setminus \mathsf{U}(R)$ for $i \in \{j+1, \ldots, n\}$. Therefore all summands on the right side belong to the ideal $R \setminus \mathsf{U}(R)$, so $f_j \in R \setminus \mathsf{U}(R)$. On the other hand $f_j = a_j + x g_j$, where $g_j \in K[x] \subset R$ and $x \in R \setminus \mathsf{U}(R)$ (so $x g_j \in R \setminus \mathsf{U}(R)$). Thus $a_j = f_j - x g_j \in R \setminus \mathsf{U}(R)$. But also $a_j \in K$ i $a_j \neq 0$, which is contradicts the result of the previous lemma. $\square$

Now we may state the final result of this section:

Theorem 11. *Let F be an algebraic function field over K, let R be such valuation ring in F that $K \subsetneq R \subsetneq F$.*

(1) *$R \setminus \mathsf{U}(R)$ is a principal ideal.*
(2) *If $R \setminus \mathsf{U}(R) = (p)$, then every element $z \in F \setminus \{0\}$ has a unique representation of the form $z = p^n u$ for some $n \in \mathbb{Z}$ and $u \in \mathsf{U}(R)$.*
(3) *R is a principal ideal domain.*

Proof. (1): Suppose that $R \setminus \mathsf{U}(R)$ is not principal and fix an element $0 \neq x_1 \in R \setminus \mathsf{U}(R)$. Since $R \setminus \mathsf{U}(R) \neq (x_1)$, there exists $x_2 \in R \setminus \mathsf{U}(R) \setminus (x_1)$. Thus $x_2 x_1^{-1} \notin R$ (otherwise $x_2 \in x_1 R = (x_1)$), so $x_1 x_2^{-1} = (x_2 x_1^{-1})^{-1} \in R \setminus \mathsf{U}(R)$, hence $x_1 \in x_2 R \setminus \mathsf{U}(R)$. By induction we may pick the infinite sequence $x_1, x_2, \ldots$ of the elements of the ideal $R \setminus \mathsf{U}(R)$ such that $x_i \in x_{i+1} R \setminus \mathsf{U}(R)$ for $i \in \mathbb{N}$, which is a contradiction with the previous lemma.

(2): Fix $z \in F$. Since $z \in R$ or $z^{-1} \in R$, we may assume that $z \in R$. If $z \in \mathsf{U}(R)$, then we may take $n = 0$ and $u = z$. So we may restrict ourselves to the case when $z \in R \setminus \mathsf{U}(R)$. Since $R \setminus \mathsf{U}(R) = (p)$, we get $z = p^l x$ for $x \in R$. Observe, that the sequence $z, t^{l-1}, t^{l-2}, \ldots, t$ satisfies the assumptions of the previous lemma, so its length is bounded. Let $k = \max\{l : z = p^l x, x \in R\}$ - we may assume that $z = p^k x$. It remains to check whether $x \in \mathsf{U}(R)$ is true - otherwise $x \in R \setminus \mathsf{U}(R) = (p)$, so

$x = py$ for some $y \in R$, so $z = p^{k+1}y$, which contradicts the definitoon of k. It is trivial to check that such presentation is unique.

(3): We shall show that if $(0) \neq I \lhd R$, then there exists $n \in \mathbb{N}$ such that $I = (p^n)$. Fix an ideal I, where we may assume that $I \subsetneq R$ - otherwise we take $n = 0$ and get $R = (1)$. Since $R \setminus \mathsf{U}(R)$ is a maximal ideal, we see that $I \subset R \setminus \mathsf{U}(R)$. Note that if $z \in I$ and $z = p^k u$, $k > 0$, $u \in \mathsf{U}(R)$ is the decomposition obtained in (2), then $p^k = zu^{-1} \in I$, so the set:

$$\{k : p^k \in I\}$$

is non-empty. As a subset of the set of positive integers it has the smallest element, say n. We shall see that $I = (p^n)$.

Obviously ($\supset$), because $p^n \in I$. To proove ($\subset$) fix $z \in I$ and let $z = p^k u$, $k > 0$, $u \in \mathsf{U}(R)$. Thus $k \geq n$, that is $p^{k-n} \in R$. Hence $z = p^k u = p^n p^{k-n} u \in (p^n)$. □

Thus we have described almost all valuations in the field $F(X)$. If $v : F(X) \to G \cup \{\infty\}$ is a non-trivial valuation such that $v(a) = 0$ for $a \in F$ and R is its valuation ring, then $F \subsetneq R \subsetneq F(X)$. So R is a principal ideal domain and hence it is a discrete valuation ring in its field of fractions. Since v is non-trivial, we may assume that $X \in R$, so $F[X] \subset R$ and $F(X)$ must be the field of fractions for R (as the smallest field containing $F[X]$ and - consequently - R). Since v is discrete, $G \cong \mathbb{Z}$ and v could be described as in the theorem 8.

3. Baer-Krull correspondence

Recall that a **semiordering** of a field F is the subset $S \subset F$ such that:

(1) $\bigwedge_{p_1,p_2 \in S} p_1 + p_2 \in S$,
(2) $\bigwedge_{p \in S} \bigwedge_{a \in F} a^2 \cdot p \in S$,
(3) $S \cap -S = \{0\}$,
(4) $\bigwedge_{a \in F} a \in S \vee -a \in S$.

For the given semiordering S of the field F we define:

$$a \geq_S b \text{ iff. } a - b \in S$$

and

$$a >_S b \text{ iff. } \sim b \geq_S a.$$

Clearly the relation $\geq \subset F \times F$ satisfies:

(1) $\bigwedge_{a \in F} a \geq a$,
(2) $\bigwedge_{a,b \in F} a \geq b \wedge b \geq a \Rightarrow a = b$,
(3) $\bigwedge_{a,b,c \in F} a \geq b \wedge b \geq c \Rightarrow a \geq c$,
(4) $\bigwedge_{a,b \in F} a \geq b \vee b \geq a \vee a = b$,
(5) $\bigwedge_{a,b,c \in F} a \geq b \Rightarrow a + c \geq b + c$,
(6) $\bigwedge_{a,b,c \in F} a \geq b \Rightarrow a \cdot c^2 \geq b \cdot c^2$,
(7) $\sim 0 \geq 1$

if and only if the set $S = \{a \in F : a \geq 0\}$ is a semiordering. Similarly, an **ordering** is the set $P \subsetneq F$ such that:

(1) $\bigwedge_{p_1,p_2 \in P} p_1 + p_2 \in P$,
(2) $\bigwedge_{p_1,p_2 \in P} p_1 \cdot p_2 \in P$,
(3) $\bigwedge_{a \in A} a \in P \vee -a \in P$,
(4) $P \cap -P = \{0\}$.

We denote:

$$a \geq_P b \text{ if } a - b \in P.$$

and - as above - observe that the relation $\geq \subset F \times F$ satisfies:

(1) $\bigwedge_{a \in F} a \geq a$,
(2) $\bigwedge_{a,b \in F} a \geq b \wedge b \geq a \Rightarrow a = b$,
(3) $\bigwedge_{a,b,c \in F} a \geq b \wedge b \geq c \Rightarrow a \geq c$,
(4) $\bigwedge_{a,b \in F} a \geq b \vee b \geq a \vee a = b$,
(5) $\bigwedge_{a,b,c \in F} a \geq b \Rightarrow a + c \geq b + c$,
(6) $\bigwedge_{a,b \in F} \bigwedge_{F \ni c \geq 0} a \geq b \Rightarrow a \cdot c \geq b \cdot c$,

if and only if the set $P = \{a \in F : a \geq 0\}$ is an ordering. Instead of speaking of the set P we shall often speak of the set $P^* = P \setminus \{0\}$. Clearly P is an ordering if and only if the set P^* satisfies:

(1) $\bigwedge_{p_1,p_2 \in P^*} p_1 + p_2 \in P^*$,
(2) $P^* \cap -P^* = \emptyset$,
(3) $\bigwedge_{a \in U(F)} a \in P^* \vee -a \in P^*$,
(4) $\bigwedge_{p_1,p_2 \in P^*} p_1 \cdot p_2 \in P^*$.

Similar conditions can be written for P^* in order for P to be a semiordering. Therefore we shall often confuse the notion of P and P^*. Let:

$$Y_F = \{P \subset U(F) : P \text{ is a semiordering in } F\},$$

$$X_F = \{P \subset U(F) : P \text{ is an ordering in } F\}.$$

In the sets Y_F and X_F define the **Harrison sets:**

$$H(a) = \{P \in Y_F : a \in P\},$$

$$H_X(a) = H(a) \cap X_F$$

and introduce a topology in Y_F by taking $H(a)$'s as subbasis sets. The topology in X_F is the topology induced from Y_F, since $X_F \subset Y_F$.

Let F be a fields, $\geq$ a semiordering in F, $v : F \to G \cup \{\infty\}$ a valuation. The valuation v is said to be **compatible** with the semiordering $\geq$ if:

$$\bigwedge_{a,b \in F} 0 < a \leq b \Leftrightarrow v(a) \geq v(b).$$

A subset $A \subset F$ is said to be **symmetric** if:

$$\bigwedge_{a \in F} a \in A \Rightarrow -a \in A.$$

A symmetric subset $A \subset F$ is **convex** (with respect to the semiordering $\geq$) if:

$$\bigwedge_{a,b \in F} 0 \leq a \leq b \wedge b \in A \Rightarrow a \in A.$$

Lemma 4. *Let F be a field , $v : F \to G \cup \{\infty\}$ a valuation, $\geq$ an ordering. The following are equivalent:*

(1) *$\geq$ is compatible with v,*
(2) *A_v is convex with respect to $\geq$,*
(3) *M_v is conver with respect to $\geq$,*
(4) $\bigwedge_{a \in F} 0 \leq a \wedge a \in M_v \Rightarrow a < 1$.

Proof. (1) $\Rightarrow$ (2): Let $a, b \in F$ be such that $0 \leq a \leq b$. Suppose that $b \in A_v$. Then $v(a) \geq v(b) \geq 0$, so $a \in A_v$.

(2) $\Rightarrow$ (3): Let $a, b \in F$ be such that $0 \leq a \leq b$. Suppose that $b \in M_v$. Clearly $\frac{1}{b} \leq \frac{1}{a}$ and $\frac{1}{b} \neq A_v$. Thus $\frac{1}{a} \neq A_v$, so $a \in M_v$.

(3) $\Rightarrow$ (4): Let $a \in F$ be such that $0 \leq a$. Suppose that $a \in M_v$. If $1 \leq a$ then $1 \in M_v$ - a contradiction.

(4) $\Rightarrow$ (1): Let $a, b \in F$ be such that $0 \leq a$. Suppose that $v(a) < v(b)$. Then $0 < v(b) - v(a) = v(\frac{b}{a})$, so $\frac{b}{a} \in M_v$. Thus $\frac{b}{a} < 1$, so $b < a$. □

Denote:

$$X_F^v = \{P \in X_F : P \text{ is compatible with } v\},$$

$$Y_F^v = \{P \in Y_F : P \text{ is compatible with } v\}.$$

Remark 1. *X_F^v and Y_F^v are closed subsets of X_F and Y_F, respectively, for all v.*

Proof. We shall show that $Y_F \setminus Y_F^v$ is open. Fix $P \in Y_F \setminus Y_F^v$. Then for some $a, b \in F$ we have $a \in P$, $b - a \in P$ and $v(a) < v(b)$. Thus $H(a) \cap H(b-a)$ is an open neighbourhood of P. Moreover, $H(a) \bigcap H(b-a) \bigcap Y_F^v = \emptyset$ - otherwise, if $Q \in H(a) \bigcap H(b-a) \bigcap Y_F^v$, then $a \in Q$, $b - a \in Q$, so $v(a) \geq v(b)$ - a contradiction.

Since X_F is a closed subset of Y_F, X_F^v is a closed subset of X_F. □

Let F be a field, $v : F \to G \cup \{\infty\}$ a valuation. A **semisection** is a mapping $s : G \to U(F)$ such that:

(1) $s(0) = 1$,
(2) $v(s(g)) = g$,
(3) $\frac{s(g_1+g_2)}{s(g_1) \cdot s(g_2)} \in U(F)^2$, that is $s(g_1 + g_2) \equiv s(g_1) \cdot s(g_2) mod U(F)^2$.

A **section** is a mapping $s : G \to U(F)$ such that:

(1) $s(0) = 1$,
(2) $v(s(g)) = g$,
(3) $s(g_1 + g_2) = s(g_1) \cdot s(g_2)$.

Remark 2. *Let F be a field, $v : F \to G \cup \{\infty\}$ a valuation, $s : G \to U(F)$ a semisection.*

(1) $s(g_1 + g_2) \in U(F)^2$,
(2) $g_1 - g_2 = g_3 + g_3 \Rightarrow \frac{s(g_1)}{s(g_2)} \in U(F)^2$, *that is* $g_1 \equiv g_2 mod 2G \Rightarrow s(g_1) \equiv s(g_2) mod U(F)^2$.

As an example consider $G = \mathbb{Z}$ and $v : F \to \mathbb{Z} \cup \{\infty\}$ a valuation. Let $p \in U(F)$ be such that $v(p) = 1$. Then $s : \mathbb{Z} \to U(F)$ gicen by:

$$s(n) = p^n$$

is a semisection.

Theorem 12. *Let F be a field, $v : F \to G \cup \{\infty\}$ a valuation. Then there exists a semisection for the valuation v.*

Proof. Since v is a surjection, for every $g \in G$ there exists $a_g \in U(F)$ such that $v(a_g) = g$. Observe that $2G = \{g + g : g \in G\}$ is a subgroup of the group G which - since G is commutative - is normal. Consider the group $G/2G$. It can be viewed as a vector space over the field $\mathbb{F}_2$. Let $B \subset G$ be such subset that $\{g + 2G : g \in B\}$ is a basis for $G/2G$. Thus for all $g \in G$ there exist $g_1, \ldots, g_n \in B$ such that:

$$g = g_1 + \ldots + g_n + g' + g'.$$

Define $s : G \to U(F)$ by:

$$s(g) = a_{g_1} \cdot \ldots \cdot a_{g_n} \cdot (a_{g'})^2.$$

It is trivial to check that s is a semisection for v. □

Lemma 5. *Let F be a field, $v : F \to G \cup \{\infty\}$ a valuation, F_v a residue field for v, $s : G \to U(F)$ a semisection for v. Then every semiordering $P \in Y_F^v$ induces a pair of mappings $\phi_P : G/2G \to Y_{F_v}$ and $\sigma_P : G/2G \to \{-1, 1\}$ given by:*

$$\bigwedge_{g \in G} \sigma_P(g + 2G) \cdot s(g) \in P$$

and

$$\bigwedge_{g \in G} \bigwedge_{b \in U(A_v)} b + M_v \in \phi_P(g + 2G) \Leftrightarrow b \cdot s(g) \cdot \sigma_P(g + 2G) \in P.$$

Proof. It suffices to verify that ϕ_P and σ_P are well-defined. If $g' + 2G = g'' + 2G$, $g', g'' \in G$, then $g' \equiv g'' mod 2G$, so by the previous remark $s(g') \equiv s(g'') mod U(F)^2$, so σ_P is well-defined.

If $b' + M_v = b'' + M_v$, $b', b'' \in U(A_v)$, then $b' = b'' + m$ for some $m \in M_v$. Suppose that $b' s(g) \sigma_P(g + 2G) \in P$. We have that:

$$\begin{aligned} v(b' s(g) \sigma_P(g + 2G)) &= v(b') + v(\pm s(g)) = v(b') + v(s(g)) = \\ &= v(b') + g = v(b'' + m) + g = \min\{v(b''), v(m)\} + g < \\ &< v(m) + g = v(m s(g) \sigma_P(g + 2G)) \end{aligned}$$

because if $b'' \in U(A_v)$ and $m \in M_v$ then $v(b'') = 0$, $v(m) > 0$. Since v is compatible with P:

$$b'' s(g) \sigma_P(g + 2G) = (b' - m) s(g) \sigma_P(g + 2G) \in P.$$

Thus ϕ_P is well-defined. It is easy to check that $\phi_P(g + 2G)$ is a semiordering. □

Corollary 1. *Let F be a field, $v : F \to G \cup \{\infty\}$ a valuation. If $Y_F^v \neq \emptyset$, then $A_v = F$ or F is formally real.*

Proof. Let $P \in Y_F^v$. By the previous lemma $\phi_P(0 + 2G)$ is a semiordering in F_v. Thus F_v is ordered and hence real (by the Zorn's lemma every semiordering can be extended to a ordering). □

Lemma 6. *Let F be a field, $v : F \to G \cup \{\infty\}$ a valuation, F_v a real closed residue field, $s : G \to U(F)$ a semisection of v. Every pair of functions $\phi : G/2G \to Y_{F_v}$ and $\sigma : G/2G \to \{-1, 1\}$ induces a semiordering $\phi^\sigma \in Y_F^v$ given by:*

$$\bigwedge_{a \in U(F)} a \in \phi^\sigma \Leftrightarrow \frac{a}{s(v(a))} \sigma(v(a) + 2G) + M_v \in \phi(v(a) + 2G).$$

Proof. We shall show that:

$$\bigwedge_{a, b \in \phi^\sigma} a + b \in \phi^\sigma.$$

Fix elements $a, b \in \phi^\sigma$. Then

$$\frac{a}{s(v(a))}\sigma(v(a)+2G)+M_v \in \phi(v(a)+2G),$$

$$\frac{b}{s(v(b))}\sigma(v(b)+2G)+M_v \in \phi(v(b)+2G).$$

If $v(a) = v(b)$ then:

$$\begin{aligned}(\frac{a}{s(v(a))}\sigma(v(a)+2G)+M_v)+(\frac{b}{s(v(b))}\sigma(v(b)+2G)+M_v) &= \\ = \quad \frac{a+b}{s(v(a))}\sigma(v(a)+2G)+M_v &\in \phi(v(a)+2G),\end{aligned}$$

so $\frac{a+b}{s(v(a))} \in U(A_v)$ - otherwise $0 \in \phi(v(a)+2G)$ which contradicts the definition of a semiordering. Thus $v(a+b) = v(s(v(a))) = v(a)$ and hence $\frac{a+b}{s(v(a+b))}\sigma(v(a+b)+2G)+M_v \in \phi(v(a+b)+2G)$, that is $a+b \in \phi^\sigma$.

If $v(a) < v(b)$, then $v(a \pm b) = \min\{v(a), v(b)\} = v(a)$. Moreover, $v(\pm\frac{b}{s(v(a))}) = v(b) - v(s(v(a))) = v(b) - v(a) > 0$, so $\pm\frac{b}{s(v(a))} \in M_v$. Thus $\frac{a\pm b}{s(v(a\pm b))}\sigma(v(a\pm b)+2G)+M_v \in \phi(v(a\pm b)+2G)$, so $a \pm b \in \phi^\sigma$.

If $v(b) < v(a)$, then $v(a+b) = \min\{v(a), v(b)\} = v(b)$, Moreover, $v(\frac{a}{s(v(b))}) = v(a) - v(s(v(b))) = v(a) - v(b) > 0$, so $\frac{a}{s(v(b))} \in M_v$. Thus $\frac{a+b}{s(v(a+b))}\sigma(v(a+b)+2G)+M_v \in \phi(v(a+b)+2G)$, so $a+b \in \phi^\sigma$.

Considering the case when $v(a) < v(b)$ we proved that ϕ^σ is compatible with v. Now we shall show that:

$$\bigwedge_{a\in\phi^\sigma}\bigwedge_{b\in U(F)} ab^2 \in \phi^\sigma.$$

Fix $a \in \phi^\sigma$ and $b \in U(F)$. Then:

$$\frac{a}{s(v(a))}\sigma(v(a)+2G)+M_v \in \phi(v(a)+2G).$$

Moreover:

$$\begin{aligned}(\frac{b}{s(v(b))}+M_v)^2 &= \frac{b^2}{s(v(b))^2}+M_v = \frac{b^2 s(v(a))}{s(v(b))^2 s(v(s))s(0)}+M_v = \\ &= \frac{b^2 s(v(a))}{s(v(a)+v(b)+v(b))}+M_v = \frac{b^2 s(v(a))}{s(v(ab^2))}+M_v\end{aligned}$$

and

$$v(ab^2)+2G = v(a)+v(b)+v(b)+2G = v(a)+2G$$

so

$$\frac{ab^2}{s(v(ab^2))}\sigma(v(ab^2)+2G)+M_v \in \phi(v(ab^2)+2G)$$

which implies that $ab^2 \in \phi^\sigma$.

Finally, it remains to show that:

$$\bigwedge_{a\in U(F)} a \in \phi^\sigma \vee -a \in \phi^\sigma,$$

but this is clear by the definition of $\phi(g+2G)$, $g \in G$. $\square$

Now we may state the main result of the current stection:

Theorem 13 (Baer-Krull)**.** *Let F be a field, $v : F \to G \cup \{\infty\}$ a valuation, F_v a real closed residue field, $s : G \to U(F)$ a semisection of v. The constructions described in lemmas 5 and 6 establish a bijective correspondence between the set Y_F^v and the set $\{\phi : \phi : G/2G \to Y_{F_v}\} \times \{\sigma : \sigma : G/2G \to \{-1,1\}, \sigma(0+2G) = 1\}$, that is $(\phi_P)^{\sigma_P} = P$ and $(\phi_{\phi^\sigma}, \sigma_{\phi^\sigma}) = (\phi, \sigma)$.*

Proof. We shall show that for all $P \in Y_F^v$ $(\phi_P)^{\sigma_P} = P$. Fix $P \in Y_F^v$. Let ϕ_P and σ_P be defined as in the lemma 5. Then for $a \in U(F)$:

$$\begin{aligned} a \in (\phi_P)^{\sigma_P} \quad &\Leftrightarrow \quad \frac{a}{s(v(a))}\sigma_P(v(a)+2G) + M_v \in \phi_P(v(a)+2G) \Leftrightarrow \\ &\Leftrightarrow \quad \frac{a}{s(v(a))}\sigma_P(v(a)+2G)s(v(a))\sigma_P(v(a)+2G) = \\ &= \quad a\sigma_P(v(a)+2G)^2 = a \in P. \end{aligned}$$

We shall show that $(\phi_{\phi^\sigma}, \sigma_{\phi^\sigma}) = (\phi, \sigma)$. Let $\phi : G/2G \to Y_{F_v}$ and $\sigma : G/2G \to \{-1,1\}$ be such that $\sigma(0+2G) = 1$. Let ϕ^σ be defined as in the lemma 6. Then:

$$\begin{aligned} &\frac{\sigma(g+2G)s(g)}{s(v(\sigma(g+2G)s(g)))}\sigma(v(\sigma(g+2G)s(g))+2G) + M_v = \\ = \quad &\frac{\sigma(g+2G)s(g)}{s(g)}\sigma(g+2G) + M_v = 1 + M_v = \\ = \quad &1^2 + M_v \in \phi(v(\sigma(g+2G)s(g))+2G). \end{aligned}$$

Thus $\sigma(g+2G)s(g) \in \phi^\sigma$ and hence $\sigma = \sigma_{\phi^\sigma}$. Moreover, for $g \in G$ and $b \in U(A_v)$ we have:

$$\begin{aligned} &b + M_v \in \phi_{\phi^\sigma}(g+2G) \Leftrightarrow bs(g)\sigma_{\phi^\sigma}(g+2G) \Leftrightarrow bs(g)\sigma(g+2G) \in \phi^\sigma \Leftrightarrow \\ \Leftrightarrow \quad &\frac{bs(g)\sigma(g+2G)}{s(v(bs(g)\sigma(g+2G)))}\sigma(v(bs(g)\sigma(g+2G))+2G) + M_v = \\ = \quad &\frac{bs(g)\sigma(g+2G)}{s(v(b)+g)}\sigma(v(b)+g+2G) + M_v = \\ = \quad &b\sigma(g+2G)^2 + M_v = \\ = \quad &b + M_v \in \phi(v(bs(g)\sigma(g+2G))+2G) = \phi(g+2G). \end{aligned}$$

□

Let G be any group and K a field. A **character** of the group G in the field K is a homomorphism $\chi : G \to U(K)$.

Theorem 14. *Let F be a field, $v : F \to G \cup \{\infty\}$ a valuation, F_v a real closed residue field, $s : G \to U(F)$ a semisection of v. Use the notation from lemmas 5 and 6. Then $P \in X_F^v$ if and only if $\phi_P : G/2G \to X_{F_v}$ is constant and σ_P is a character on $G/2G$.*

Proof. ($\Rightarrow$): Let $P \in X_F^v$. Then for $g_1, g_2 \in G$:

$$\begin{aligned} \sigma_P((g_1+2G)+(g_2+2G)) = 1 \quad &\Leftrightarrow \quad s(g_1+g_2) \in P \Leftrightarrow s(g_1)s(g_2 \in P) \Leftrightarrow \\ &\Leftrightarrow \quad s(g_1), s(g_2) \in P \vee s(g_1)s(g_2) \notin P \Leftrightarrow \\ &\Leftrightarrow \quad \sigma_P(g_1+2G)\sigma_P(g_2+2G) = 1. \end{aligned}$$

So - since the value set of σ_P consists of only two elements - this implies that σ_P is a homomorphism. Next, for $b \in U(A_v)$ by the lemma 5:

$$b + M_v \in \phi_P(g + 2G) \Leftrightarrow bs(g)\sigma_P(g + 2G) \in P$$

and

$$b + M_v \in \phi_P(0 + 2G) \Leftrightarrow bs(0)\sigma_P(0 + 2G) = b \in P.$$

Since $\sigma_P(g + 2G)s(g) \in P$ and P is an ordering, this yields:

$$b + M_v \in \phi_P(g + 2G) \Leftrightarrow bs(g)\sigma_P(g + 2G) \in P \Leftrightarrow b \in P \Leftrightarrow b + M_v \in \phi_P(0 + 2G),$$

so ϕ_P is constant.

($\Leftarrow$): Let σ_P be a character, let ϕ_P be constant. We shall show that for $a_1, a_2 \in P$ $a_1a_2 \in P$. Indeed, by lemma 6 for $a_1, a_2 \in P$:

$$\frac{a_i}{s(v(a_i))}\sigma_P(v(a_i) + 2G) + M_v \in \phi_P(v(a_i) + 2G) = \phi_P(0 + 2G), i \in \{1, 2\}.$$

Since $s(v(a_1a_2)) \equiv s(v(a_1))s(v(a_2))modU(F)^2$, we have:

$$\begin{aligned}
&\frac{a_1a_2}{s(v(a_1a_2))}\sigma_P(v(a_1a_2) + 2G) + M_v = \\
&= f^2\frac{a_1a_2}{s(v(a_1))s(v(a_2))}\sigma_P((v(a_1) + 2G) + (v(a_2) + 2G)) + M_v = \\
&= [\frac{a_1}{s(v(a_1))}\sigma_P(v(a_1) + 2G) + M_v] \cdot [\frac{a_2}{s(v(a_2))}\sigma_P(v(a_2) + 2G) + M_v] \cdot \\
&\quad \cdot [f^2 + M_v] \in \phi_P(0 + 2G) = \phi_P(v(a_1a_2) + 2G).
\end{aligned}$$

□

Corollary 2. *Let F be a field, $v : F \to G \cup \{\infty\}$ a valuation, F_v a real closed residue field. Then $X_F^v \neq \emptyset$ and every semiordering $P \in Y_F^v$ is non-Archimedean.*

Proof. Since F_v is real, we have that $X_{F_v} \neq \emptyset$. Fix an arbitrary constant function $\phi : G/2G \to X_{F_v}$ and a constant character $\phi_P : G/2G \to \{-1, 1\}$. By the Baer-Krull theorem and the previous theorem, $X_F^v \neq \emptyset$. Fix $P \in Y_F^v$ and $g \in G$ such that $g < 0$. Let $a \in P$ be such that $v(a) = g$. Since F_v is real, $v(n) = 0$ for $n \geq 1$. Since P is compatible with v, we have that, as $a \in P$ and $v(a) < v(n)$ - $a, n \in P$. Thus P cannot be Archimedean. □

Corollary 3. *Let F be a field, $v : F \to G \cup \{\infty\}$ a valuation, F_v a real closed residue field. Then $X_F^v = Y_F^v$ if and only if*

$$|G/2G| = 2 \textit{ and } |X_{F_v}| = 1$$

or

$$|G/2G| = 1 \textit{ and } X_{F_v} = Y_{F_v}.$$

Proof. Let $s : G \to U(F)$ be a semisection. Observe that the mapping $\sigma : G/2G \to \{-1, 1\}$ such that $\sigma(0+2G) = 1$ is a character if and only if $|G/2G| \leq 2$. Moreover, if $|X_{F_v}| = 1$, then $|Y_{F_v}| = 1$. In this case every mapping $\phi : G/2G \to X_{F_v}$ is constant. Thus our statement follows from the Baer-Krull theorem and the previous result. □

Let $(F, \geq)$ be a field with semiordering $\geq$, let $A, B \subset F$. We say that A is **cofinite** in B with respect to $\geq$ it $A \subset B$ and

$$\bigwedge_{b \in B} \bigvee_{a \in A} b \leq a.$$

Remark 3. *Let $(F, \geq)$ be a field with semiordering $\geq$, let $F_0 \subset F$ be a subfield. The set*

$$A^{\geq}_{F_0} = \{a \in F : |a| \geq b \text{ for some } b \in F_0\}$$

is the smallest convex set B such that F_0 is cofinite in B.

Lemma 7. *Let $(F, \geq)$ be a field with semiordering $\geq$, $F_0 \subset F$ be a subfield. Then $A^{\geq}_{F_0}$ is a valuation ring in F convex with respect to $\geq$.*

Proof. Obviously $\mathbb{Q} \subset A^{\geq}_{F_0}$ (where $\mathbb{Q}$ means the field isomorphis with the rationals) and $A^{\geq}_{F_0}$ is closed with respect to addition and subtraction. We shall show that if $a \in A^{\geq}_{F_0}$, then $a^2 \in A^{\geq}_{F_0}$. Indeed, fix $a \in A^{\geq}_{F_0}$. There exists $b \in F_0$ such that $|a| \leq b$. We may assume that $1 < b$. Thus $|a| \leq b^2$ and $|a^2| = |a|^2 \leq b^4 \in F_0$, so $a^2 \in A^{\geq}_{F_0}$.

That mens that - since $(\frac{a+b}{2})^2 - (\frac{a-b}{2})^2 = ab$ - $A^{\geq}_{F_0}$ is also closed under multiplication. It remains to show that it is a valuation ring. Fix $a \in F$ and suppose that $a \notin A^{\geq}_{F_0}$. Again, we may assume that $1 < a$. Then $0 < a^{-1} < 1$, so $a^{-1} \in A^{\geq}_{F_0}$. □

Theorem 15. *The topological space X_F of orderings is the sum of the Archimedean orderings and the sets X^v_F, where $v : F \to G \cup \{\infty\}$ are such valuations that F_v are real.*

Proof. Let $\geq$ be a non-Archimedean ordering. By the previous lemma $A^{\geq}_{\mathbb{Q}}$ is a valuation ring different from F. Let $v : F \to G \cup \{\infty\}$ be such valuation that $A_v = A^{\geq}_{\mathbb{Q}}$. Since A_v is convex with respect to $\geq$, v is compatible with $\geq$. Thus $\geq \in X^v_F$ and by the previous corollary F_v is a real field. □

4. Orderings of $\mathbb{Q}(X)$

By the Baer-Krull theorem we know that orderings in $\mathbb{Q}(X)$ arise from the orderings in the residue fields associated with valuations on $\mathbb{Q}(X)$. By the previous examples we know that the residue fields associated with valuations on $\mathbb{Q}(X)$ are just the algebraic number fields $\mathbb{Q}(\alpha)$. Next, by the Artin-Schreier theorem, $\mathbb{Q}(\alpha)$ is an ordered field if and only if $\mathbb{Q}(\alpha) \subset \mathbb{R}$, that is if $\alpha \in \mathbb{R}$. Therefore $\mathbb{R}$ is the real closure for all every field $\mathbb{Q}(\alpha)$. We shall describe orderings in $\mathbb{Q}(\alpha)$ in more details.

First, observe that orderings on $\mathbb{Q}(\alpha)$ are in bijective correspondence with $\mathbb{Q}$-embeddings of $\mathbb{Q}(\alpha)$ into $\mathbb{R}$. Indeed, let $\sigma : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{R}$ be an $\mathbb{Q}$-embedding. Then σ defines an ordering $P = \sigma^{-1}(\mathbb{R}^2)$. If $\sigma_1 \neq \sigma_2$ then $\sigma_1^{-1}(\mathbb{R}^2) \neq \sigma_2^{-1}(\mathbb{R}^2)$. For if $\sigma_1^{-1}(\mathbb{R}^2) = \sigma_2^{-1}(\mathbb{R}^2)$ then $\sigma_2 \circ \sigma_1^{-1} : \sigma_1(\mathbb{Q}(\alpha)) \to \sigma_2(\mathbb{Q}(\alpha))$ defines an order-preserving isomorphism between two ordered fields lying in the same real closed field, so - by the uniqueness of the real closure - $\sigma_2 \circ \sigma_1 = id$ and thus $\sigma_1 = \sigma_2$ - a contradiction. Thus our correspondence is one-to-one. To show its surjectivity, let P' be an ordering of $\mathbb{Q}(\alpha)$. Let $\overline{\mathbb{Q}(\alpha)}$ be a real closure of $\mathbb{Q}(\alpha)$ extending the ordering P'. By the uniqueness of the real closure, there exists an isomorphism $\sigma^* : \overline{\mathbb{Q}(\alpha)} \to \mathbb{R}$, which induces an embedding $\sigma : \mathbb{Q}(\alpha) \to \mathbb{R}$ such that $\sigma = \sigma^*|_{\mathbb{Q}(\alpha)}$. Clearly $P' = \sigma^{-1}(\mathbb{R}^2)$.

Next, let f be a minimal polynomial for α. We shall show that embeddings of $\mathbb{Q}(\alpha)$ into $\mathbb{R}$ are in bijective correspondence with real roots of f. Indeed, let $\sigma : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{R}$ be an embedding. f - as a polynomial with coefficients in $\mathbb{Q}$ - may be viewed as a polynomial in $\mathbb{R}[X]$ as well. Clearly $f(\sigma(\alpha)) = 0$, so $\sigma(\alpha)$ is a root of f. Such correspondence is one-to-one: assume that $\sigma_1(\alpha) = \sigma_2(\alpha)$. Clearly $\mathbb{Q}(\alpha) = \{a_0 + a_1\alpha + \ldots + a_{m-1}\alpha^{m-1} : a_i \in \mathbb{Q}\}$, where $m = [\mathbb{Q}(\alpha) : \mathbb{Q}]$, so we may write:

$$\begin{aligned}\sigma_1(a_0 + a_1\alpha + \ldots + a_{m-1}\alpha^{m-1}) &= a_0 + a_1\sigma_1(\alpha) + \ldots + a_{m-1}\sigma_1(\alpha)^{m-1} = \\ &= a_0 + a_1\sigma_2(\alpha) + \ldots + a_{m-1}\sigma_2(\alpha)^{m-1} = \\ &= \sigma_2(a_0 + a_1\alpha + \ldots + a_{m-1}\alpha^{m-1}),\end{aligned}$$

which implies that $\sigma_1 = \sigma_2$. To chow that the correspondence is surjective, fix a root β of $f \in \mathbb{R}$. If $m = [\mathbb{Q}(\alpha) : \mathbb{Q}]$ then for all $l < m$ the elements $1, \beta, \ldots, \beta^{l-1}$ are algebraically independent; indeed, suppose that for some $b_1 \in \mathbb{Q}$:

$$b_0 + b_1\beta + \ldots + b_{l-1}\beta^{l-1} = 0.$$

Then $b_0 + b_1X + \ldots + b_{l-1}X^{l-1} \in \{h \in \mathbb{Q}[X] : h(\beta) = 0\}$ and $I = \{h \in \mathbb{Q}[X] : h(\beta) = 0\}$ is an ideal such that $f \in I$. Sinve $\mathbb{Q}[X]$ is a principal ideal domain, $I = (g)$ for some $g \in \mathbb{Q}[X]$ and since $b_0 + b_1X + \ldots + b_{l-1}X^{l-1} \in I$ we have that $\deg g \leq l - 1 < m - 1$. Moreover, $g|f$, which is a contradiction, since f is irreducible. Thus $1, \beta, \ldots, \beta^{l-1}$ are algebraically independent and $\sigma : \mathbb{Q}(\alpha) \to \mathbb{R}$ given by $\sigma(a_0 + \ldots + a_{m-1}\alpha^{m-1}) = a_0 + \ldots + a_{m-1}\beta^{m-1}$ is a well-defined embedding.

Now we shall describe orderings in the field $\mathbb{Q}(X)$ associated with valuations. As we already know, the valuations on $\mathbb{Q}(X)$ are in bijective correspondence with irreducible polynomials $f \in \mathbb{Q}[X]$ and the element ∞. Let $f \in \mathbb{Q}(X)$ be an irreducible polynomial, let $n = \deg f$. Denote by $v_{X,f}$ the valuation associated with f, namely $v_{X,f} : \mathbb{Q}(X) \to \mathbb{Z} \cup \{\infty\}$ given by:

$$v_{X,f}(g) = \begin{cases} \infty & \text{if } g = 0, \\ n_f & \text{if } g = u\prod_{h\in\mathbb{P}} h(X)^{n_h}, n_h \in \mathbb{Z}, u \in \{-1, 1\}. \end{cases}$$

where $\mathbb{P}$ denotes the set of all irreducible polynomials in $\mathbb{Q}(X)$. Clearly the valuation ring associated with $v_{X,f}$ is:

$$A_{v_{X,f}} = \{\frac{p}{q} : f \nmid q\}$$

with the only maximal ideal given by:

$$M_{v_{X,f}} = \{\frac{p}{q} : f \nmid q, f|q\}.$$

Observe that the residue field associated with such valuation satisfies:

$$F_{v_{X,f}} = A_{v_{X,f}}/M_{v_{X,f}} \cong \mathbb{Q}[X]/(f).$$

Indeed, consider the homomorphism $\Phi : \mathbb{Q}[X] \to A_{v_{X,f}}/M_{v_{X,f}}$ given by:

$$\Phi(g) = g + M_{v_{X,f}}.$$

Clearly, the kernel of Φ is the ideal generated by f. Moreover, Φ is surjective: if $g \in A_{v_{X,f}}$, we can write $g = \frac{p(X)}{q(X)}$ with $p, q \in \mathbb{Q}[X]$ such that $p \nmid q$. Thus there are $r, s \in \mathbb{Q}[X]$ with $r(X)f(X) + s(X)q(X) = 1$, therefore

$$g = 1 \cdot g = \frac{r(X)p(X)}{q(X)}f(X) + s(X)p(X)$$

and $g + M_{v_{X,f}} = s(X)p(X) + M_{v_{X,f}}$ is in the image of Φ. Thus Φ induces an isomorphism $\widehat{\Phi} : \mathbb{Q}[X]/(f) \to A_{v_{X,f}}/M_{v_{X,f}}$ given by:

$$\widehat{\Phi}(g + (f)) = g + M_{v_{X,f}}.$$

Next, suppose that $\alpha_1, \ldots, \alpha_m$ are all real roots of f. We shall show that the residue field $\mathbb{Q}[X]/(f)$ can be mapped isomorphically onto each of the fields $\mathbb{Q}(\alpha_i)$. Indeed, fix $i \in \{1, \ldots, m\}$ and define the mapping $\Psi : \mathbb{Q}[X] \to \mathbb{Q}(\alpha_i)$ by:

$$\Psi(g) = g(\alpha_i).$$

Clearly Ψ is surjective and its kernel is the ideal generated by f, so Ψ induces an isomorphism $\widehat{\Psi} : \mathbb{Q}[X]/(f) \to \mathbb{Q}(\alpha_i)$ given by:

$$\widehat{\Psi}(g + (f)) = g(\alpha_i).$$

We shall describe the orderings of $\mathbb{Q}(\alpha_i)$ more precisely. We know that - via the Baer-Krull correspondence - those orderings play an essential role in building orderings compatible with $v_{X,f}$. From the previous remarks we know that the residue field of $v_{X,f}$ is isomorphic to $\mathbb{Q}(\alpha_i)$ and that $\mathbb{Q}(\alpha_i)$ has m orderings corresponding to variuos embedings of $\mathbb{Q}(\alpha_i)$ into $\mathbb{R}$, each determined by the element α_i and one of the real roots α_j of f. Let P_{α_i,α_j} denote the ordering of $\mathbb{Q}(\alpha_i)$ derived from the embedding which maps α_i onto α_j. We shall show that:

$$(\mathbb{Q}(\alpha_i), P_{\alpha_i,\alpha_j}) \text{ is order-isomorphic to } (\mathbb{Q}(\alpha_j), P_{\alpha_j,\alpha_j}).$$

Clearly the fields $\mathbb{Q}(\alpha_i)$ and $\mathbb{Q}(\alpha_j)$ are isomorphic and the isomorphism $\Gamma : \mathbb{Q}(\alpha_i) \to \mathbb{Q}(\alpha_j)$ is given by:

$$\Gamma(a_0 + \ldots + a_{n-1}\alpha_i^{n-1}) = a_0 + \ldots + a_{n-1}\alpha_j^{n-1}.$$

Let $\sigma_{\alpha_i,\alpha_j} : \mathbb{Q}(\alpha_i) \hookrightarrow \mathbb{R}$ and $\sigma_{\alpha_j,\alpha_j} : \mathbb{Q}(\alpha_j) \hookrightarrow \mathbb{R}$ be the embeddings given by:

$$\sigma_{\alpha_i,\alpha_j}(a_0 + \ldots + a_{n-1}\alpha_i^{n-1}) = a_0 + \ldots + a_{n-1}\alpha_j^{n-1}$$

and

$$\sigma_{\alpha_j,\alpha_j} = id.$$

Then $P_{\alpha_i,\alpha_j} = \sigma^{-1}_{\alpha_i,\alpha_j}(\mathbb{R}^2)$ and $P_{\alpha_j,\alpha_j} = \sigma^{-1}_{\alpha_j,\alpha_j}(\mathbb{R}^2)$ and we have to show that $\Gamma(P_{\alpha_i,\alpha_j}) = P_{\alpha_j,\alpha_j}$. This is clear; $(\subset)$: let $a_0 + \ldots + a_{n-1}\alpha_i^{n-1} \in P_{\alpha_i,\alpha_j}$, that is $a_0 + \ldots + a_{n-1}\alpha_j^{n-1} \in \mathbb{R}^2$ which means that $\Gamma(a_0 + \ldots + a_{n-1}\alpha_i^{n-1}) = a_0 + \ldots + a_{n-1}\alpha_j^{n-1} \in P_{\alpha_j,\alpha_j}$. $(\supset)$: let $a_0 + \ldots + a_{n-1}\alpha_j^{n-1} \in P_{\alpha_j,\alpha_j}$, that is $a_0 + \ldots + a_{n-1}\alpha_j^{n-1} \in \mathbb{R}^2$, which means that $a_0 + \ldots + a_{n-1}\alpha_i^{n-1} \in P_{\alpha_i,\alpha_j}$ while $\Gamma(a_0 + \ldots + a_{n-1}\alpha_i^{n-1}) = a_0 + \ldots + a_{n-1}\alpha_j^{n-1}$.

Thus instead of considering $\mathbb{Q}(\alpha_i)$ with the ordering P_{α_i,α_j}, we may consider the field $\mathbb{Q}(\alpha_j)$ with the usual ordering induced from the reals. Therefore the set $Y_{F_{v_{X,f}}}$ may be viewed as a family of the fields $\{\mathbb{Q}(\alpha_i) : i \in \{1, \ldots, m\}\}$ with the usual orderings.

Next, the semisection $s : \mathbb{Z} \to \mathbb{Q}(X) \setminus \{0\}$ of $v_{X,f}$ is clearly given by: $s(n) = f^n$ and each function $\phi : \mathbb{Z}/2\mathbb{Z} \to Y_{F_{v_{X,f}}}$ is constant, that is we may say that there are m such functions $\phi_1, \ldots, \phi_m$, each ϕ_i - in view of the previous remarks - taking $\mathbb{Q}(\alpha_i)$ with usual ordering as the only value. As we know there are 2 characters $\sigma : \mathbb{Z}/2\mathbb{Z} \to \{-1, 1\}$, namely σ_1 and σ_2 defined by:

x	$0 + 2\mathbb{Z}$	$1 + 2\mathbb{Z}$
σ_1	1	-1

x	$0 + 2\mathbb{Z}$	$1 + 2\mathbb{Z}$
σ_2	1	1

.

Thus f induces $2m$ orderings, namely:

$$\phi_1^{\sigma_1}, \dots, \phi_m^{\sigma_1}, \phi_1^{\sigma_2}, \dots, \phi_m^{\sigma_2}.$$

We shall describe such orderings in terms of polynomials. Let $g \in \mathbb{Q}[X]$ and suppose that $g = f^{v(g)} \cdot h$. We have:

$$\begin{aligned} g \in \phi_i^{\sigma_1} \Leftrightarrow & \frac{g}{s(v(g))}\sigma_1(v(g)+2\mathbb{Z}) + M_{v_{X,f}} \in P_{\alpha_i,\alpha_i} \Leftrightarrow \\ \Leftrightarrow \quad & h\sigma_1(v(g)+2\mathbb{Z}) + (f) \in P_{\alpha_i,\alpha_i} \Leftrightarrow \\ \Leftrightarrow \quad & h\sigma_1(v(g)+2\mathbb{Z})(\alpha_i) \in P_{\alpha_i,\alpha_i} \Leftrightarrow \\ \Leftrightarrow \quad & h\sigma_1(v(g)+2\mathbb{Z})(\alpha_i) \in \mathbb{R}^2 \Leftrightarrow \\ \Leftrightarrow \quad & (h(\alpha_i) \in \mathbb{R}^2 \wedge v(g) \text{ - even }) \vee (-h(\alpha_i) \in \mathbb{R}^2 \wedge v(g) \text{ - odd }). \end{aligned}$$

Similarly:

$$g \in \phi_i^{\sigma_1} \Leftrightarrow h(\alpha_i) \in \mathbb{R}^2.$$

This describes all orderings associated with polynomials.

Let $v_{X,\infty} : \mathbb{Q}(X) \to \mathbb{Z} \cup \{\infty\}$ be the remaining valuation of $\mathbb{Q}(X)$ given by:

$$v_{X,\infty}(g) = \begin{cases} \infty, & \text{if } g = 0, \\ \deg q - \deg p, & \text{if } g = \frac{p(X)}{q(X)}, p, q \in \mathbb{Q}[X] \setminus \{0\}. \end{cases}$$

The valuation ring for such valuation is defined by:

$$A_{v_{X,\infty}} = \{\frac{p}{q} : \deg p \leq \deg q\}$$

and the only maximal ideal is:

$$M_{v_{X,\infty}} = \{\frac{p}{q} : \deg p < \deg q\}.$$

The residue field $F_{v_{X,\infty}} = A_{v_{X,\infty}}/M_{v_{X,\infty}}$ is isomorphic to $\mathbb{Q}$ and the isomorphism $\widehat{\Phi} : A_{v_{X,\infty}}/M_{v_{X,\infty}} \to \mathbb{Q}$ is given by

$$\widehat{\Phi}(\frac{a_n X^n + \ldots + a_0}{b_n X^n + \ldots + b_0}) + M_{v_{X,\infty}} = \frac{a_n}{b_n}.$$

The semisection $s : \mathbb{Z} \to \mathbb{Q}(X) \setminus \{0\}$ is clearly given by:

$$s(n) = \frac{1}{X^n}.$$

Since the degree of such valuation is 1, the residue field is just $\mathbb{Q}$. $\mathbb{Q}$ has only one ordering, so the set $Y_{F_{v_{X,\infty}}}$ consists of only one element, which may be viewed as the field $\mathbb{Q}$ with the usual ordering and there is only one constant function $\phi : \mathbb{Z}/2\mathbb{Z} \to Y_{F_{v_{X,\infty}}}$. Thus - since we have two characters $\sigma : \mathbb{Z}/2\mathbb{Z} \to \{-1, 1\}$ - there are only two orderings associated with this valuation, namely

$$\phi^{\sigma_1}, \phi^{\sigma_2}.$$

Let us see how this ordering works with polynomials. Let $g = a_n X^n + \ldots + a_0 \in \mathbb{Q}[X]$. Then:

$$\begin{aligned} g \in \phi^{\sigma_1} \Leftrightarrow & \frac{g}{s(v(g))}\sigma_1(v(g)+2\mathbb{Z}) + M_{v_{X,\infty}} \in \mathbb{Q}^+ \Leftrightarrow \\ \Leftrightarrow \quad & a_n \cdot \sigma_1(v(g)+2\mathbb{Z}) \in \mathbb{Q}^+ \Leftrightarrow \\ \Leftrightarrow \quad & (a_n > 0 \wedge n \text{ - even }) \vee (a_n < 0 \wedge n \text{ - odd }). \end{aligned}$$

Similarly:

$$g \in \phi^{\sigma_2} \Leftrightarrow a_n > 0$$

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