A survey on multirings, hyperrings and hyperfields

Paweł Gładki *†

Institute of Mathematics,

University of Silesia,

ul. Bankowa 14,

Katowice, Poland, 40-007

pawel.gladki@us.edu.pl

Abstract

In these notes we outline some of the main concepts of the theory of rings and fields with multivalued addition. We focus on examples and applications, often referring to other published sources whenever it comes to more detailed discussion on some of the theoretical concepts.

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1 Introduction

There has been considerable interest recently in hyperfields, hyperrings and multirings. This interest derives not so much from the actual objects themselves as from the success achieved in using these objects to understand and explain other objects and phenomena. Hyperfields and hyperrings arise in

*Corresponding author: phone +48 32 359 2228, fax +48 32 258 2976

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the study of the algebraic structure of the adèle class space of a global field and in exploring the deeper relationship between algebraic number fields and algebraic function fields [3], [4]. Hyperfields occur naturally in the context of quadratic form theory and spaces of orderings [15], Milnor K-theory [16], tropical geometry [24], commutative algebras over fields with semi-linear homomorphisms, abelian groups with injective homomorphisms, as well as non-desarguesian plane projective geometries [3], whose examples include Moulton plane, projective planes of order 9 with 91 points and 91 lines, Hughes plane, Moufang planes and André planes; see [25] for an extensive survey on the topic. The latter ones fit into the propositional formulation of quantum theory as given by Piron: if the plane is coordinated by Cayley numbers, the units can be identified with a base of infinitesimal generators of the Lorentz group, as shown in [23]. Multirings are considered in [15], and spaces of signs, also known as abstract real spectra, objects which arise naturally in the study of constructible sets in real geometry [1], [13], are shown to be multirings of a particular sort.

Hyperrings and hyperfields were introduced first by Krasner [10], [11] in connection with his work on valued fields. Multirings and multifields were introduced later and independently in [15]. All of these objects are very natural and very useful, although they are not at all widely known. In the present survey we outline some of the main concepts of the theory, and invite the reader to enter the fascinating world of applications of multirings, hyperrings, and hyperfields. In Section 2 we give some basic definitions and discuss some of the issues that arise when choosing axioms of the theory. In Section 3 we outline the theory of hyperfield extensions and show how it is essentially tied to projective geometry; the case of Desarguesian geometries corresponds to hyperfield extensions of a particularly pleasant form. In Section 4 we recall how some phenomena in quantum physics can be interpreted through orthocomplemented lattices and, in turn, can be rephrased in the language of hyperrings via connection between complemented modular lattices and projective geometries. In Section 5 we elaborate on applications of multivalued addition to tropical geometry. In Section 6 we give an example of the hyperring of adele classes of a global field.

2 Basic definitions

A *multiring* [15] is a system $(A, +, \cdot, -, 0, 1)$ where A is a set, $+$ is a multivalued binary operation on A, i.e., function from $A \times A$ to the set of all subsets of A, $\cdot$ is a binary operation on A, $- : A \to A$ is a function, and $0, 1$ are elements of A such that

I. $(A, +, -, 0)$ is a canonical hypergroup, terminology as in [18], i.e.,

(**M1**) $c \in a + b \Rightarrow a \in c + (-b)$,
(**M2**) $a \in b + 0$ iff $a = b$,
(**M3**) $(a + b) + c = a + (b + c)$,
(**M4**) $a + b = b + a$; and

II. $(A, \cdot, 1)$ is a commutative monoid, i.e.,

(**M5**) $ab = ba$,
(**M6**) $(ab)c = a(bc)$,
(**M7**) $a1 = a$ for all $a \in A$.

III. Moreover, we require that

(**M8**) $a0 = 0$ for all $a \in A$, and
(**M9**) $a(b + c) \subset ab + ac$.

A *multifield* is a multiring with $1 \neq 0$ such that every non-zero element has a multiplicative inverse. *Hyperrings* and *hyperfields* are defined by Krasner in [10] and [11]. A hyperring is a multiring which also satisfies the second half of the distributive property, i.e., $ab + ac \subseteq a(b + c)$. For a multifield, the second half of the distributive property is automatic from the first half, i.e., hyperfields and multifields are the same thing. Let us start with some examples of discussed structures.

Example 1. *The simplest example of a hyperfield is* $Q_2 = \{-1, 0, 1\}$. *Here addition and multiplication are defined in the obvious way, by interpreting* 1 *to mean positive,* -1 *to mean negative, and* 0 *to mean zero, i.e.,* $0 \cdot x = x \cdot 0 = 0$, $(1) \cdot (1) = (-1) \cdot (-1) = 1$, $(1) \cdot (-1) = (-1) \cdot (1) = -1$, $x + 0 = 0 + x = x$, $1 + 1 = 1$, $(-1) + (-1) = -1$, *and* $1 + (-1) = (-1) + 1 = \{-1, 0, 1\}$ *(since positive plus negative is indeterminate).*

There are many interesting examples of multirings which are not hyperrings. The real reduced multirings constructed in [15] are typically not hyperrings. Let us have a look at a relatively obvious example.

Example 2. *Let V be an algebraic set in R^n where R is a real closed field, and let A denote the coordinate ring of V, i.e., the ring of all polynomial functions $f : V \to R$. Define an equivalence relation $\sim$ on A by declaring $f \sim g$ to mean that f, g have the same sign ($+$,$-$, or 0) at each point of V. The set of equivalence classes is the real reduced multiring denoted by $Q_{\text{red}}(A)$ in [15]. It is made into a multiring as follows: Denote the equivalence class of f by $\overline{f}$. Define $\overline{f} \in \overline{g} + \overline{h}$ to mean $\exists\ f', g', h' \in A$ such that $f' = g' + h'$, $\overline{f'} = \overline{f}$, $\overline{g'} = \overline{g}$ and $\overline{h'} = \overline{h}$. Define $\overline{g}\overline{h} = \overline{gh}$, $-\overline{f} = \overline{-f}$, $0 = \overline{0}$, and $1 = \overline{1}$. The multiring $Q_{\text{red}}(A)$ is not a hyperring if $\dim(V) \geq 1$. For example, if $V = R$ (so $n = 1$ and A is the polynomial ring $R[x]$), and a, b, c and d are the classes of the polynomials $x, x, 1$ and x^2+x^3, respectively, then $d \in ab+ac$ but $d \notin a(b+c)$. This is because d is positive for x close to zero, $x \neq 0$, but any element of $a(b+c)$ is negative for x close to zero, $x < 0$. We also use the fact that x^3 and x have the same sign.*

If S, T are subsets of a multiring A then $S+T :=$ the union of the sets $x+y$, $x \in S$, $y \in T$, and $ST := \{xy \mid x \in S, y \in T\}$. Also, $S-T := S+(-T)$, where $-T := \{-y \mid y \in T\}$. $\sum S$ denotes the union of all finite sums $x_1 + \cdots + x_n$, $x_1, \ldots, x_n \in S$, $n \geq 1$.

We refer the reader to [15] for basic terminology and basic facts concerning multirings and hyperfields. We recall parts of this. A *multiring homomorphism* from A to B, where A and B are multirings, is a function $f : A \to B$ satisfying $f(a+b) \subseteq f(a)+f(b)$, $f(ab) = f(a)f(b)$, $f(-a) = -f(a)$, $f(0) = 0$, and $f(1) = 1$. Note that in the ring case some of these axioms are consequences of the other ones, for example $f(0) = 0$ is a consequence of $f(1) = 1$ and $f(a + b) = f(a) + f(b)$, but this is no longer true in the multiring case, as it might happen that $\{0\} \subsetneq 1-1$. There are more interesting phenomena concerning multiring homomorphisms – for example it is no longer equivalent for a homomorphism to be injective and to have a zero kernel.

Example 3. *Let V be an algebraic set in R^n where R is a real closed field, let A denote the coordinate ring of V, and consider the multiring $Q_{\text{red}}(A)$ defined above. The natural homomorphism $A \to Q_{\text{red}}(A)$ given by $f \mapsto \overline{f}$ has kernel equal to $\{0\}$, but is clearly not injective, for example $\overline{f^3} = \overline{f}$, but rarely $f^3 = f$.*

We say A is *strongly embedded* in B by a multiring homomorphism $i : A \to B$ if i is injective and, for all $a, b, c \in A$, $i(c) \in i(a) + i(b) \Rightarrow c \in a + b$.

This does not necessarily mean that $i(A)$ is a submultiring of B. There is no requirement that $i(a)+i(b)$ is a subset of $i(A)$!

Ideals and multiplicative sets are defined in an obvious way. If S is a multiplicative set in A and I is an ideal of A, then one can form the localization $S^{-1}A$ and the factor multiring A/I, and there are natural multiring homomorphisms $A \to S^{-1}A$ and $A \to A/I$. The principal ideal of A generated by $x \in A$ is the set $\sum Ax :=$ the union of all sets of the form $a_1x + \cdots + a_nx$, $a_i \in A$, $n \geq 1$. If A is a hyperring, this coincides with the set $Ax := \{ax \mid a \in A\}$.

We denote the hyperfield of fractions of a multidomain D by $\mathrm{ff}(D)$, i.e., $\mathrm{ff}(D) := (D\backslash\{0\})^{-1}D$. It is important to realize that the natural multiring homomorphism $D \to \mathrm{ff}(D)$ is not injective in general:

Example 4. *Let V be an algebraic set in R^n where R is a real closed field, let A denote the coordinate ring of V, and consider the multiring $Q_{\mathrm{red}}(A)$ defined above. Assume further that V is irreducible. Then A is a domain and $D = Q_{\mathrm{red}}(A)$ is a multidomain. For example, suppose V is the elliptic curve $y^2 = x(x+1)^2$ in $\mathbb{R}^2$. Since $(x+1)x$ and $(x+1)x^2$ have the same sign on V, $\overline{x+1}\cdot\overline{x} = \overline{x+1}\cdot\overline{x^2}$ in D. Since $\overline{x+1} \neq 0$ in D, this implies $\overline{x} = \overline{x^2}$ in $\mathrm{ff}(D)$. But $\overline{x} \neq \overline{x^2}$ in D (since x and x^2 have different signs at the isolated point).*

To make things more complicated, even in the case when $D \to \mathrm{ff}(D)$ turns out to be injective, D might not be strongly embedded into $\mathrm{ff}(D)$:

Example 5. *In the above example take $V = \mathbb{R}$ (so A is the polynomial ring $\mathbb{R}[x]$). In this case the homomorphism $D \to \mathrm{ff}(D)$ is injective. Since $\overline{1} \in \overline{1}+\overline{1}$ holds in D and $\overline{x^2} = \overline{1}$ holds in $\mathrm{ff}(D)$ (since $\overline{x^3} = \overline{x}$ holds in D and $\overline{x} \neq 0$), we see that $\overline{x^2} \in \overline{1}+\overline{1}$ holds in $\mathrm{ff}(D)$. But $\overline{x^2} \in \overline{1}+\overline{1}$ cannot hold in D (because x vanishes at the origin but 1 is positive at the origin). Thus the embedding $D \to \mathrm{ff}(D)$ is not a strong embedding.*

If S is a multiplicative subset in a multiring A, there is another construction one can perform, which we denote by $A/_mS$ and refer to as *the quotient construction* [11], [15, Example 2.6]. $A/_mS$ is the set of equivalence classes with respect to the equivalence relation $\sim$ on A defined by $a \sim b$ iff $as = bt$ for some $s \in S$. The operations on $A/_mS$ are the obvious ones induced by the corresponding operations on A. Denote by $\overline{a}$ the equivalence class of a. Then $\overline{a} \in \overline{b}+\overline{c}$ iff $as \in bt + cu$ for some $s, t, u \in S$, $\overline{a}\overline{b} = \overline{ab}$, $-\overline{a} = \overline{-a}$.

Also, $0 = \overline{0}$, and $1 = \overline{1}$. A special case of this construction appears already in quadratic form theory.

Example 6. *Let F be a field of characteristic $\neq 2$, $F \neq \mathbb{F}_3, \mathbb{F}_5$, and consider the multifield $Q(F) := F/mF^{*2}$, where F^{*2} denotes the subgroup $\{a^2 : a \in F^*\}$ of the multiplicative group $F^* = F \setminus \{0\}$ of F . ($Q(\mathbb{F}_3)$ and $Q(\mathbb{F}_5)$ are also defined, but the definition is not quite the same.) $Q(F)$ is nothing more or less than the special group of F [5] (also called the quadratic form scheme of F [14]) with zero adjoined. If $a_i \in Q(F)$, $a_i \neq 0$, $i \in \{1, ..., n\}$, then $a_1 + + a_n$ is precisely the value set of the associated diagonal quadratic form. If F^2 has finite index in F^*, then $Q(F) = F^*/F^{*2} \cup \{0\}$ has order $2^n + 1$, where $2^n = (F^* : F^{*2})$. The possible structures of $Q(F)$ (as F varies) have been computed for $n \leq 5$; see [14]. For $n = 0$ there is just one possibility, namely $Q_1 := \{0, 1\}$ with addition and multiplication defined by $x \cdot 0 = 0 \cdot x = 0$, $1 \cdot 1 = 1$, $0 + x = x + 0 = x$, $1 + 1 = \{0, 1\}$. For $n = 1$ there are 3 possibilities (the multifield Q_2 defined earlier and 2 others). For $n = 2$ (resp., 3, 4, 5), there are 6 (resp., 17, 51, 155) possibilities.*

For a multiring A, we are interested in the set $\{x^k \mid x \in A\}$, which we denote by A^k for short, so $\sum A^k$ denotes the union of all finite sums $x_1^k + \cdots + x_n^k$, $x_1, \ldots, x_n \in A$, $n \geq 1$. We are especially interested in the case where $k = 2^\ell$.

Let A be a multiring, $\ell \geq 1$ and integer. A *preordering of level ℓ of A* is a subset T of A satisfying $T + T \subseteq T$, $TT \subseteq T$ and $a^{2^\ell} \in T$ for all $a \in A$. One can readily see that a preordering of level 1 is simlply what was defined as a preordering in [15]. We say the preordering T of A is *proper* if $-1 \notin T$. A *T-module of A* is a subset M of A satisfying $M + M \subseteq M$, $TM \subseteq M$, and $1 \in M$.

For a prime ideal $\mathfrak{p}$ of A, the *residue hyperfield* of A at $\mathfrak{p}$ is defined to be $\mathrm{ff}(A/\mathfrak{p})$, the hyperfield of fractions of the multidomain $A/\mathfrak{p}$. For a preordering T of level ℓ of A, we denote by $T_\mathfrak{p}$ the extension of T to $\mathrm{ff}(A/\mathfrak{p})$. The preordering $T_\mathfrak{p}$ is proper iff the prime ideal $\mathfrak{p}$ is T-convex.

By an *ordering of level ℓ* of a multiring A we mean a pair $(\mathfrak{p}, P)$ where $\mathfrak{p}$ is a prime ideal of A and P is an ordering of level ℓ on $\mathrm{ff}(A/\mathfrak{p})$. The prime ideal $\mathfrak{p}$ is called the *support* of $(\mathfrak{p}, P)$. We denote by $\mathrm{Sper}_\ell(A)$ the set of all orderings of level ℓ of A and by X_T the set of all orderings $(\mathfrak{p}, P)$ of level ℓ of A with P lying over $T_\mathfrak{p}$.

3 Desarguesian geometries and hyperfield extensions

By a *projective plane* we mean a set, whose elements are called *points*, together with a family of subsets called *lines*, satisfying the following axioms:

(**P1**) Any two distinct points belong to exactly one line;

(**P2**) Any two distinct lines meet in exactly one point;

(**P3**) There exists a quadrilateral: a set of four points, no three on any line.

Perhaps the most familiar example is the real projective plane $\mathbb{P}^2(\mathbb{R})$, whose points are the lines through the origin in Euclidean 3-space and whose lines are planes in 3-space. Of course the projective plane $\mathbb{P}^2(F)$ over any field F will also be a projective plane. The smallest projective plane is $\mathbb{P}^2(\mathbb{F}^2)$. It has 7 points and 7 lines, and is often called the Fano plane.

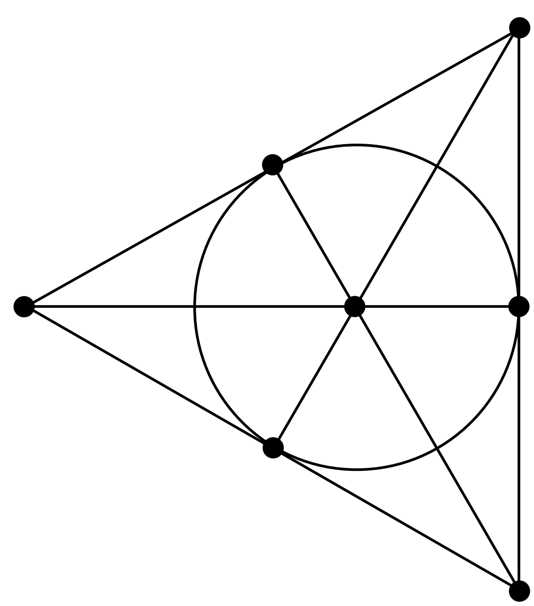

Figure 1: The Fano plane of 7 points and 7 lines

If there are exactly $q+1$ points on any (hence every) line, we say that the plane has order q. A plane of order q has q^2+q+1 points, and also q^2+q+1 lines. Of course, $\mathbb{P}^2(\mathbb{F}_q)$ has order q. It is conjectured that the order q of a finite projective plane must be a prime power; this is known only for $q \leq 11$.

A projective plane is the same as a 2−dimensional projective geometry. By a *d−dimensional projective geometry*, we mean a set (of points) $\mathcal{P}$, together with a family of subsets (lines) $\mathcal{L}$ satisfying the following axioms:

(**PG1**) Two distinct points $x, y \in \mathcal{P}$ lie on exactly one line; this line will be denoted by $L(x, y)$;

(**PG2**) If a line meets two sides of a triangle, not at their intersection, then it also meets the third side;

(**PG3**) Every line contains at least 3 points;

(**PG4**) The set of all points is spanned by $d + 1$ points, and no fewer.

For reasons that will become apparent soon, we also introduce a stronger version of the axiom (**PG3**):

(**PG3'**) Every line contains at least 4 points.

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The feature that makes projective planes more complicated than higher dimensional projective geometries is that Desargues Theorem need not hold. We say that two triangles are *perspective from a point* P (resp., *from a line* L) if their corresponding vertices are on lines through P (resp., edges meet on L).

Theorem 7 (Desargues)**.** *Let F be any field (or division ring). Two triangles in $\mathbb{P}^d(F)$ are perspective from a point if and only if they are perspective from a line.*

A projective geometry is said to be *Desarguesian* if whenever two triangles are perspective from a point, they are perspective from a line, and vice versa. If this property fails, it is said to be *non-Desarguesian.* Any Desarguesian projective geometry is just a projective space $\mathbb{P}^d(F)$ over a field (or division ring) F, a fact already known to Hilbert . If $d \geq 3$, every d−dimensional projective geometry is Desarguesian. The projective plane over Cayleys Octonions is non-Desarguesian. Every finite projective plane of order $q \leq 8$ is Desarguesian, and hence is isomorphic to the plane $\mathbb{P}^2(\mathbb{F}_q)$. There are three distinct non-Desarguesian planes of order 9, each consisting of 91 points and, consequently, 91 lines.

Automorphisms of a projective plane must preserve lines, so they are called *collineations.* The collineations form a group, and the geometry of the plane is reflected by the structure of this group.

There is a beautiful and really quite astonishing link between non-Desarguesian geometries and hyperfields, in particular the theory of hyperfield

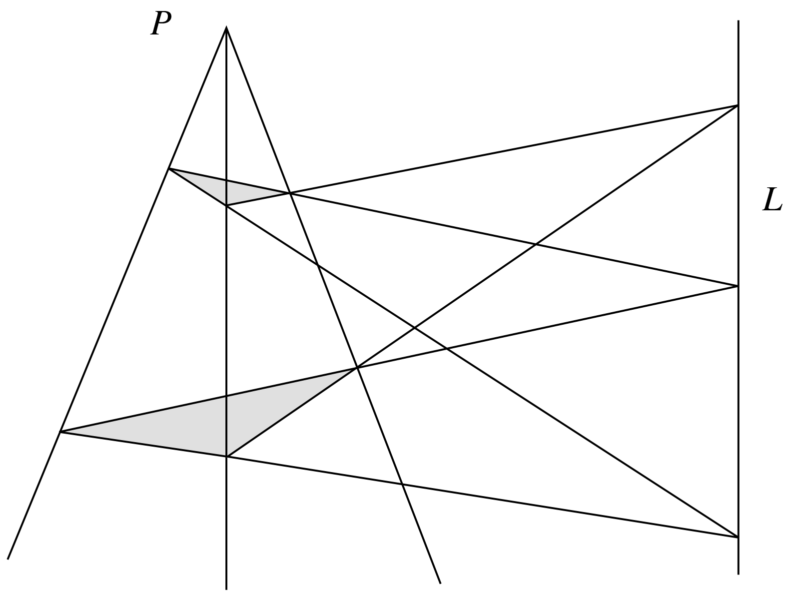

Figure 2: Desargues' Theorem

extensions, that we shall describe now in some detail. We use the terminology of a Q_1-vector space to refer to a (commutative) hypergroup V with a compatible action on Q_1. We have the following:

Theorem 8. *Let V be a Q_1-vector space, let $\mathcal{P} = V \setminus \{0\}$. Then there exists a unique geometry having $\mathcal{P}$ as its set of points and such that the line through two distinct points $x, y \in \mathcal{P}$ is given by*

$$L(x, y) = x + y \cup \{x, y\} \tag{1}$$

*This geometry, in fact, satisfies axioms (**PG1**), (**PG2**), (**PG3'**).*

*Conversely, let $(\mathcal{P}, \mathcal{L})$ be a projective geometry fulfilling the axioms (**PG1**), (**PG2**), (**PG3'**). Let $V = \mathcal{P} \cup \{0\}$ be endowed with hyperaddition having 0 as neutral element, and defined by*

$$x + y = \begin{cases} L(x, y) \setminus \{x, y\}, & \text{if } x \neq y \\ \{0, x\}, & \text{if } x = y. \end{cases} \tag{2}$$

Then V is a Q_1-vector space.

The proof is pretty straightforward and is essentially due to Prenowitz [22] and Lyndon [12]. The theorem in the quoted form is taken from [3].

Next result shows that hyperfield extensions of Q_1 correspond precisely to the Zweiseitiger Inzidenszgruppen (two-sided incidence groups) of [6]. In particular, the commutative hyperfield extensions of Q_1 are classified by projective geometries together with a simply transitive action by a commutative subgroup of the collineation group. We first recall the definition of a two-sided incidence group: let G be a group which is the set of points of a projective geometry. Then G is called a *two-sided incidence group* if the left and right translations by G are automorphisms of the geometry. We can now state the precise relation between hyperfield extensions of Q_1 and twosided incidence groups whose projective geometry satisfies the axiom (**PG3'**) in place of (**PG3**):

Theorem 9. *Let $H \supset Q_1$ be a hyperfield extension of Q_1. Let $(\mathcal{P}, \mathcal{L})$ be the associated projective geometry. Then, the multiplicative group of H endowed with the geometry $(\mathcal{P}, \mathcal{L})$ is a two-sided incidence group fulfilling (***PG3'***).*

*Conversely, let G be a two-sided incidence group fulfil ling (***PG3'***). Then, there exists a unique hyperfield extension $H \supset Q_1$ such that $H = G \cup \{0\}$. The hyperaddition in H is defined by the rule*

$$x + y = L(x, y) \setminus \{x, y\}, \text{ for any } x \neq y \in \mathcal{P},$$

and the multiplication is the group law of G, extended by

$$0 \cdot g = g \cdot 0 = 0, \forall g \in G.$$

The case when underlying geometry is Desarguesian is especially important due to the convenience in describing the structure of the hyperfield H (see [3] and [9]):

Theorem 10. *Let $H \supset Q_1$ be a commutative hyperfield extension of Q_1. Assume that the geometry associated to the Q_1-vector space H is Desarguesian and of dimension ≥ 2. Then, there exists a unique pair (K, k) of a commutative field K and a subfield $k \subset K$ such that*

$$H = K/mk^*$$

4 Quantum logic and lattice theory

It is generally believed that the crucial difference between classical and quantum physics can be expressed in the language of mathematical logic and lattice theory. This idea was introduced in 1932 by Birkhoff and von Neumann and continued by Jauch and Piron. They postulated the existence of a certain non-classical logical system reflecting the nature of quantum phenomena and called *quantum logic*. The physical meaning of quantum logic can be illustrated on the following model. Consider a stationary beam of particles and a class of filters which can be used to select the beam. Suppose the beam passes through various sequences of filters and we observe the resulting sub-beams. Assume, however, that we possess only specific detectors. They do not yield any numerical measure of the beam intensity. They only allow the comparison of intensities: we can observe two sub-beams and recognize the "more intense" one. One can ask what sort of physics can be constructed on the basis of these experiments. The answer is that we shall arrive precisely at the "quantum logic". First we define vacuum (absence of beam): this is the beam of the smallest intensity possible. Next we discover the existence of the relations of equivalence ($\equiv$), inequality ($\leq$), and orthogonality ($\perp$) for some pairs of filters. More precisely, we call two filters a and b *equivalent* ($a \equiv b$) if the substitution of a by b (and of b by a) in any chain of filters selecting the beam does not affect the intensity of the resulting sub-beam. For two filters a, b we say that a is *contained* in b ($a \leq b$) if any beam energing from a passes through b without being partially absorbed. We call two filters a, b *orthogonal* if the successive application of a and b (and b and a) produces the vacuum. By observing the structure of the set $\mathcal{Q}$ of all known filters we notice that:

(Q1) The inequality $\leq$ is a partial order relation in $\mathcal{Q}$.

(Q2) For any $a, b \in \mathcal{Q}$ the subclass of all filters containing both a and b contains the smallest element. We call this element the *union* of a and b and we denote it by $a \vee b$. Similarly, for any $a, b \in \mathcal{Q}$ the subclass of filters contained in both a and b contains the greatest element. We call it the *intersection* of a and b and we denote it by $a \wedge b$.

(Q3) For any $a \in \mathcal{Q}$ the subclass of all filters orthogonal to a contains the greatest element a'. The correspondence $a \to a'$ obeys the rules:

(a) $I' \equiv \emptyset$,
(b) $(a')' \equiv a$,
(c) $(a \vee b)' \equiv a' \wedge b'$.

Points (Q1), (Q2), and (Q3) imply that the set $\mathcal{Q}$ with the relation $\leq$ and with the mapping $a \to a'$ is an *orthocomplemented lattice*. Recall that an orthocomplemented lattice is a partially ordered set $(\mathcal{Q}, \leq)$ such that every pair of elements a, b has a join $a \vee b$ and a meet $a \wedge b$, has the greatest element

I and the smallest element $\emptyset$, in which every element has a distinguished complement, called an orthocomplement, that behaves like the complementary subspace of a subspace in a vector space. More precisely, a complement of a is an element b such that $a \wedge b = \emptyset$ and $a \vee b = I$. Now denote by M the set of complements of elements of $\mathcal{Q}$. M is clearly a partially ordered subset of $\mathcal{Q}$, with $\leq$ inherited from $\mathcal{Q}$. For each $a \in \mathcal{Q}$, let $M_a \subset M$ be the set of complements of a. $\mathcal{Q}$ is said to be orthocomplemented if there is a function $\perp: L \to M$, called an *orthocomplementation*, whose image is written $a^{\perp}$ for any $a \in \mathcal{Q}$, such that:

(1) $a^{\perp} \in M_a$,
(2) $(a^{\perp})^{\perp} = a$, and
(3) $\perp$ is order-reversing; that is, for any $a, b \in \mathcal{Q}$, $a \leq b$ implies $b^{\perp} \leq a^{\perp}$.

The element $a^{\perp}$ is called an *orthocomplement* of a (via $\perp$).

Provided that the properties (Q1), (Q2), and (Q3) hold, we can introduce the analogy between the set of filters and a logical system as follows. We call the set $\mathcal{Q}$ the *logic of the beam of particles.* Any filter is called a *proposition.* The inequality $a \leq b$ means "a implies b". The operations $a \vee b$, $a \wedge b$ and $a \to a$ are interpreted as the alternative, conjunction and negation of the logic respectively. If the beam in question is a beam of microparticles (such as photons) the resulting logical system is non-distributive, i. e., the following distributive law does not hold:

$$a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c) \tag{3}$$

The absence of (3) implies the existence of incompatible propositions in $\mathcal{Q}$. Two propositions are called *compatible* if the smallest orthocomplemented sub-lattice of $\mathcal{Q}$ containing both a and b is distributive, otherwise they are called *incompatible*. Many authors consider the non-distributive character of quantum logic and the existence of incompatible propositions as the most important manifestation of the quantum nature of microphenomena. See [17] for more details.

A lattice is called *modular* if the following axiom holds:

$$a \vee (b \wedge c) = (a \vee b) \wedge c$$

The *length* of a partially ordered set is defined as the least upper bound of the lengths of the chains in $\mathcal{Q}$, and denoted by $l(\mathcal{Q})$. In a partially ordered set of finite length we define the *height* of an element a as the least upper bound of lengths of the chains $\emptyset = a_0 \leq a_1 \leq \ldots \leq a_h = a$ between $\emptyset$ and a. Elements of height 1 are called *points* or *atoms*, while elements of height 2 are called *lines*. The following theorem establishes a link between modular lattices and projective geometries, and is due to Birkhoff [2]

Theorem 11. *There is a one-to-one correspondence between $d-$dimensional projective geometries and simple complemented modular lattices of dimension $d+1$, $d \neq 0$. Under this correspondence, the projective geometry is the set of points and lines of the lattice.*

Here by *simple* lattice we understand a lattice without quotients; for precise definitions of quotient lattices and dimensions of lattices, we refer to Birkhoff's book [2]. This establishes the link between quantum theory and hyperfields via projective geometries.

5 Tropical geometry

In this section we provide a list of examples of hyperfields recently studied by Viro in [24] that are to be applied in tropical geometry in subsequent works by their inventor.

In the set $\mathbb{R}_+$ of non-negative real numbers, dene a multivalued addition $\triangledown$ by formula

$$a \triangledown b = \{c \in \mathbb{R}_+ : |ab| \leq c \leq a + b\}.$$

In other words, $a \triangledown b$ is the set of all real numbers c such that there exists an Euclidean triangle with sides of lengths a, b, c. The set $\mathbb{R}_+$ with the multivalued addition $\triangledown$ and usual multiplication is a hypereld. This hyperfield is called the *triangle hyperfield* and denoted by $\triangle$. Addition in $\triangle$ is obviously commutative. It is also associative. In order to prove this, just observe that both $(a \triangledown b) \triangledown c$ and $a \triangledown (b \triangledown c)$ coincide with the set of real numbers x such that there exists an Euclidean quadrilateral with sides of lengths a, b, c, x. The usual multiplication is distributive over $\triangledown$. The role of zero is played by 0. The negation $a \mapsto -a$ for $\triangledown$ is identity, as for any $a \in \mathbb{R}_+$ the only real number x such that $0 \in a \triangledown x$ is a. We note, however, that $\triangledown$ is not double-distributive, that is

$$(a \triangledown b)(x \triangledown y) \neq ax \triangledown ay \triangledown bx \triangledown by.$$

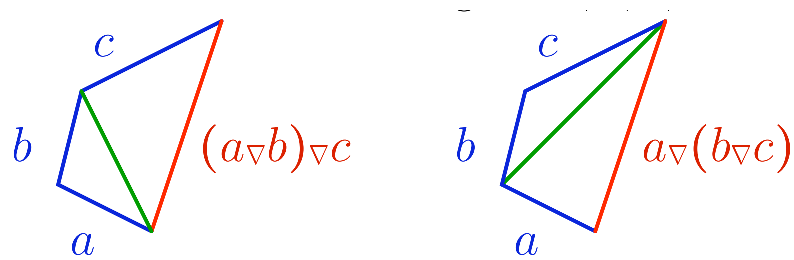

Figure 3: Associativity of $\triangledown$

Indeed, $2\triangledown 1 = [1,3]$. Therefore $(2\triangledown 1)(2\triangledown 1) = [1,3][1,3] = [1,9]$. On the other hand, $2\cdot 2\triangledown 2\cdot 1\triangledown 1\cdot 2\triangledown 1\cdot 1 = 4\triangledown 2\triangledown 2\triangledown 1$ contains 0, because there exists an isosceles trapezoid with sides $4,2,1$, and 2. In fact, $4\triangledown 2\triangledown 2\triangledown 1 = [0,9]$.

Another way of defining a hyperfield in the set $\mathbb{R}_+$ that finds applications in tropical geometry is the following one. Define addition by the formula

$$(a,b) \mapsto a \curlyvee b = \begin{cases} \max(a,b), & \text{if } a \neq b \\ \{x \in \mathbb{R}_+ : x \leq a\}, & \text{if } a = b \end{cases}$$

The multiplication is the usual multiplication of real numbers. As one can easily check, this hyperfield is doubly distributive, see Section. There is also another way to construct the same hyperfield. It is completely similar to the construction of the triangle hyperfield $\triangle$, but with the triangle inequality in the definition of the addition replaced by the non-archimedian (or ultra-) triangle inequality

$$|c| \leq \max(|a|, |b|).$$

This hyperfield is called the *ultratriangle hyperfield* and denoted by $\mathbb{Y}_\times$

The map $\log : \mathbb{R}_{>0} \to \mathbb{R}$ is naturally extended by mapping 0 to $-\infty$. The resulting map $\mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is denoted also by log. This is a bijection, and the hyperfield structure of $\mathbb{Y}_\times$ can be transferred via log to $\mathbb{R} \cup \{-\infty\}$. Denote the resulting hypereld by $\mathbb{Y}$, and call it the *tropical hyperfield*. We can describe addition explicitly as follows: the underlying set of $\mathbb{Y}$ is $\mathbb{R} \cup \{-\infty\}$, the addition is

$$(a,b) \mapsto a \curlyvee b = \begin{cases} \max(a,b), & \text{if } a \neq b \\ \{x \in \mathbb{Y} : x \leq a\}, & \text{if } a = b \end{cases}$$

the multiplication is the usual addition of real numbers extended in the obvious way to $-\infty$, the hyperfield zero is $-\infty$, the hyperfield unity is $0 \in \mathbb{R}$.

One can also transfer via the same bijection $\log : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ the structure of the triangle hyperfield $\triangle$ defined above to $\mathbb{R} \cup \{-\infty\}$. The resulting hyperfield is called the *amoeba hyperfield* and denoted by $\triangle^{\log}$. The addition in $\triangle^{\log}$ is defined by the formula

$$a \curlywedge b = \{c \in \mathbb{R} : \log(|e^a - e^b|) \leq c \leq \log(e^a + e^b)\},$$

while the multiplication in $\triangle^{\log}$ is the usual addition.

As the last example in this section, we shall define the *complex tropical hyperfield*. The tropical sum $a \smile b$ of arbitrary complex numbers a and b is dened as follows.

(1) If $|a| > |b|$, then $a \smile b = \{a\}$.

(2) If $|a| < |b|$, then $a \smile b = \{b\}$.

(3) If $|a| = |b|$ and $a+b \neq 0$, then $a \smile b$ is the set of all complex numbers which belong to the shortest arc connecting a with b on the circle of complex numbers with the same absolute value.

(4) If $a + b = 0$, then $a \smile b$ is the whole closed disk $\{c \in \mathbb{C} | c| \leq |a|\}$.

The tropical addition is commutative, $a \smile b = b \smile a$ for any $a, b \in \mathbb{C}$. This follows immediately from the denition. The zero plays the same role of the neutral element as it plays for the usual addition: $a \smile 0 = a$ for any $a \in \mathbb{C}$. Furthermore, for any complex number a there is a unique b such that $0 \in a \smile b$. This b is a. A straightforward proof that the tropical addition of complex numbers is associative is elementary, but quite cumbersome. The usual multiplication of complex numbers is distributive over the tropical addition:

$$a(b \smile c) = ab \smile ac$$

for any complex numbers a, b and c. Indeed, all the constructions and characteristics of summands involved in the denition of tropical addition are invariant under multiplication by a complex number: the ratio of absolute values of two complex numbers is preserved, an arc of a circle centered at 0 is mapped to an arc of a circle centered at 0, a disk centered at 0 is mapped to a disk centered at 0. Note, however, that the multiplication of complex numbers is not doubly distributive over the tropical addition: compare $(1 \smile i)(1 \smile -i)$ with $1 \cdot 1 \smile 1 \cdot -i \smile i \cdot 1 \smile i \cdot (-i) = 1 \smile i \smile i \smile 1$

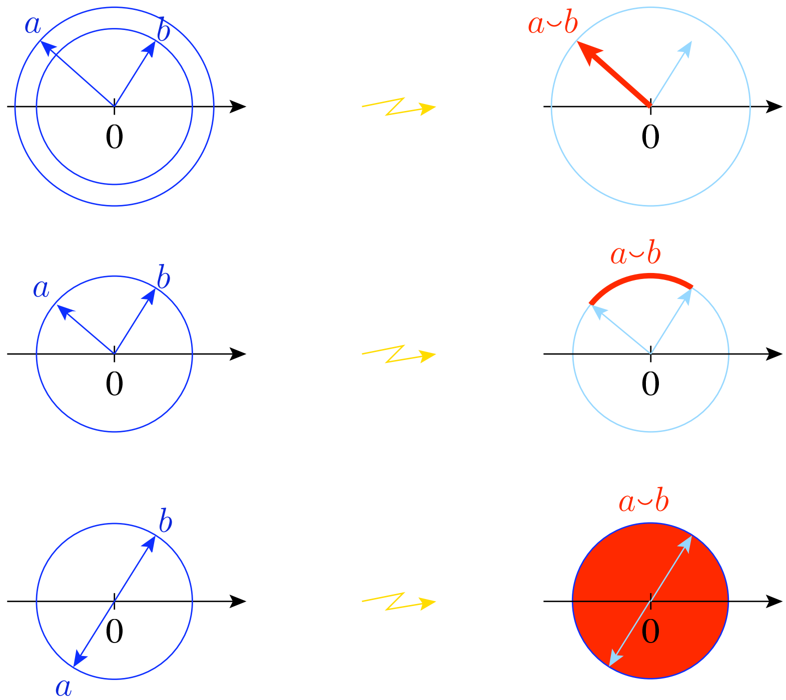

Figure 4: Tropical addition of complex numbers

6 Hyperring of ádele classes

Following [3] we give an example of the hyperring of adele classes used in number theory. Let $\widehat{Z} = \lim_{\leftarrow} \mathbb{Z}/n\mathbb{Z}$ be the inverse limit of the system of rings $\mathbb{Z}/n\mathbb{Z}$, which, by the Chinese remainder theorem, is isomorphic to the product $\prod_p \mathbb{Z}_{(p)}$ of all the rings of p-adic integers:, where p ranges over the set of prime numbers. The *ring of integral ádeles* $\mathbb{A}_{\mathbb{Z}}$ is then defined as the product $\mathbb{R} \times \widehat{\mathbb{Z}}$. Now, for any algebraic number field F, the *ring of ádeles* $\mathbb{A}_F$ is defined as the tensor product

$$\mathbb{A}_F = F \otimes_{\mathbb{Z}} \mathbb{A}_{\mathbb{Z}}.$$

The standard quotient construction can be now applied to the ring of ádeles $\mathbb{A}_F$ and its multiplicative subset F^* to obtain a hyperring $\mathbb{H}A_F = \mathbb{A}_F/mF^*$ called the *hyperring of ádele classes*. This ring has a very good algebraic structure, as shown in [3, Theorem 7.1]:

Theorem 12. *The hyperring of ádele classes is a hyperring extension of the* *hyperfield* Q_1.

7 Real reduced multirings and hyperfields. Spaces of orderings and abstract real spectra

Suppose F is a *real* hyperfield, that is a hyperfield that admits as ordering. For any proper preordering T of F, we can build the multield $Q_T(F) = F/mT$. In particular, we can build $Q_{\Sigma F^2}(F)$, which we denote simply by $Q_{red}(F)$. If T_1, T_2 are preorderings with $T_1 \subset T_2$, then the multiring homomorphism $F \to Q_{T_2}(F)$ factors through $Q_{T_1}(F)$. Consider the multield Q_2 dened earlier. $\{0, 1\}$ is an ordering of Q_2. For any ordering P of a multield F, $Q_P(F) \cong Q_2$ by a unique multiring isomorphism. Orderings of a hyperfield F correspond bijectively to multiring homomorphisms $\sigma : F \to Q_2$ via $P = \sigma^{-1}(\{0, 1\})$. $Sper(Q_{red}(F))$, the set of all orderings of the multiring $Q_{red}(F)$, is naturally identified with $Sper(F)$, the set of all orderings of the hyperfield F. $Sper(Q_T(F))$ is naturally identied with X_T, the set of all orderings of F extending T. The following is [15, Proposition 4.1]:

Theorem 13. *For a real hyperfield F the following are equivalent:*

(1) The multiring homomorphism $F \to Q_{red}(F)$ is an isomorphism.
(2) $\Sigma F^2 = \{0, 1\}$.
(3) For all $a \in F$, $a^3 = a$ and, for all $a \in F$, $a \in 1 + 1 \Rightarrow a = 1$.

A *real reduced hyperfield* is dened to be a real hyperfield satisfying the equivalent conditions of the above theorem. For any proper preordering T of a real hyperfield F, $Q_T(F)$ is a real reduced hyperfield. In particular, $Q_{red}(F)$ is a real reduced hyperfield. If $p : F_1 \to F_2$ is a multiring homomorphism of real hyperfields, then $p(\Sigma F_1^2) \subset \Sigma F_2^2$, so p induces a hyperfield homomorphism $Q_{red}(F_1) \to Q_{red}(F_2)$. In this way, Q_{red} denes a functor (a reection) from the category of real hyperfields onto the subcategory of real reduced hyperfields.

Real reduced hyperfields and spaces of orderings are essentially the same thing. If F is a real reduced multield, then the pair $(Sper(F), F)$ is a space of orderings in the terminology of [13, Section 2.1], and every space of orderings is of this form, for some unique hyperfield F. This is clear. It follows from the theory of spaces of orderings that nite real reduced hyperfields (more generally, real reduced hyperfields having finite chain length) are completely classied recursively [13, Theorem 4.22].

We can carry over a similar construction to the case of multirings. We assume that A is a multiring with $-1 \notin \Sigma A^2$ and T is a proper preordering of A. Denote the image of A in $Q_2^{X_T}$ by $Q_T(A)$. One shows that $Q_T(A)$ is a multiring strongly embedded in $Q_2^{X_T}$. The real spectrum of $Q_T(A)$ is naturally identified with X_T. We restrict our attention now to the case where $T = \Sigma A^2$ and consider the multiring homomorphism $a \mapsto \overline{a}$ from A into $Q_2^{Sper(A)}$. We denote $Q_{\Sigma A^2}(A)$ by $Q_{red}(A)$ which we refer to as the *real reduced multiring associated to* A. The multirings A such that the multiring homomorphism $a \mapsto \overline{a}$ from A onto $Q_{red}(A)$ is an isomorphism are obviously of special interest. One shows (compare [15, Proposition 7.5]):

Theorem 14. *For a multiring A with $-1 \notin \Sigma A^2$, the map $a \mapsto \overline{a}$ from A onto $Q_{red}(A)$ is an isomorphism iff A satises the following three properties (for all $a, b \in A$):*

(1) $a^3 = a$,

(2) $a + ab^2 = \{a\}$,

(3) $a^2 + b^2$ contains a unique element.

A multiring satisfying $-1 \notin \Sigma A^2$ and the equivalent conditions of the above theorem will be called a *real reduced multiring.* Real reduced multirings and abstract real spectra are the same thing: If A is a real reduced multiring, then the pair $(Sper(A), A)$ is an abstract real spectrum in the terminology of [13, Sectio 6.1], and every space of signs is of this form.

8 Orderings of higher level in multirings and hyperfields

One can generalize the concept of real reduced multirings and hyperfields to $\ell-$real reduced hyperfields, that is fields admitting orderings of level ℓ. As is explained above, real reduced hyperfields correspond to spaces of orderings, so it is natural to wonder if ℓ-real reduced hyperfields correspond to the spaces of signatures introduced in [19], [20], [21]. This question is considered in [7] and [8], and, so far, only partial answers to the question concerned are known. An example can be prodeced showing that, in fact, this is not the case, and one additional axiom, a certain symmetry property

(∗) For all odd integers $1 \leq k \leq 2^\ell$, $a \in b + c \Rightarrow a^k \in b^k + c^k$,

can be considered, which is satisfied by spaces of signatures but not by general ℓ-real reduced hyperfields. The example we mention is the following one:

Example 15. *Let* $F = \{0\} \cup \{\pm 1, \pm a, \pm a^2, \pm a^3\}$*, where* $a^4 = -1$*, with addition defined by* $1 + 1 = \{1\}$*,* $1 - 1 = F$*,* $1 + a = \{1, a, -a^2, a^3\}$*,* $1 - a = \{1, -a\}$*,* $1 + a^2 = \{1, -a, a^2\}$*,* $1 - a^2 = \{1, -a^2, a^3\}$*,* $1 + a^3 = \{1, a^3\}$*,* $1 - a^3 = \{1, -a, a^2, -a^3\}$*. This addition is extended to all of* F *in the obvious way, i.e.,* $r + s := r(1 + \frac{s}{r})$ *if* $r, s \neq 0$*. Then* F *is a* $3-$*real reduced hyperfield whose ordering* $\{0, 1\}$ *does not come from a signature.*

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