Category of hypermodules with multi-valued morphisms

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Abstract

In this paper we work exclusively in the setup of the laboratory of a utility muffin research kitchen. Reaching for an oversized chromed spoon we gather an intimate quantity of dried muffin remnants and, brushing our scapulars aside, we proceed to dump these into the inside of our shirts

Keywords: multirings, hyperrings, hyperfield, hypermodules.

1 Hypermodules

Hyperrings and hyperfields were introduced first by Krasner [3], [4] in connection with his work on valued fields. Multirings were introduced later and independently in [5] (see also [6]). All of these objects are very natural and very useful, although they are not at all widely known. For what we are doing here we recall briefly the notion of a hyperring. A **hyperring** [5] is a system $(R, +, \cdot, -, 0, 1)$ where R is a set, $+$ is a multivalued binary operation on R, i.e., a function from $R \times R$ to the set of all subsets of R, $\cdot$ is a binary operation on R, $- : R \to R$ is a function, and $0, 1$ are elements of R such that, for all $r, s, t \in R$:

(HR1) $r \in s + t \Rightarrow s \in r + (-t)$,

(HR2) $r \in s + 0$ iff $r = s$,

(HR3) $(r + s) + t = r + (s + t)$,

(HR4) $r + s = s + r$,

(HR5) $(rs)t = r(st)$,

(HR6) $rs = sr$,

(HR7) $r1 = r$,

(HR8) $r0 = 0$,

(HR9) $r(s + t) = rs + rt$.

We define hypermodules in a usual way: for a given hyperring R, a $R-$hypermodule is a system $(A, +, \cdot, -, 0)$, where A is a set, $+$ is a multivalued binary operation on A, $- : A \to A$ is a function, $\cdot : R \times A \to A$ is an external multiplication, and $0 \in A$ is an element such that, for all $a, b, c \in A$ and $r, s \in R$:

(HM1) $c \in a + b \Rightarrow a \in c + (-b)$,

(HM2) $a \in b + 0$ iff $a = b$,

(HM3) $(a + b) + c = a + (b + c)$,

(HM4) $a + b = b + a$,

(HM5) $(rs)a = r(sa)$,

(HM6) $1_R a = a$,

(HM7) $r0_A = 0_A$ and $0_R a = 0_A$,

(HM8) $r(a + b) = ra + rb$,

(HM9) $(r + s)a = ra + sa$.

Here, to avoid confusion, 0_R and 1_R were used to denote the 0 and 1 elements in the hyperring R, whilst $0_A = 0$ is the zero element of A. We note that a structure $(A, +, -, 0)$ that satisfies **(HM1) - (HM4)** will be called a (commutative) **hypergroup.**

Remark: Just as is the case with ordinary modules, we check that

(1) $(-r)a = -(ra) = r(-a)$,

(2) $(nr)a = n(ra) = r(na)$,

for all $r \in R$, $a \in A$, and $n \in \mathbb{N}$, with the usual meaning of nx as $\underbrace{x + x + \ldots + x}_{n}$.

Examples: (1) Just as rings are special cases of hyperrings, $R-$modules over are examples of $R-$hypermodules.

(2) $\mathbb{Z}-$hypermodules are just hypergroups.

(3) For a hyperring R an ideal is a subset $I \subset R$ such that $I + I \subset I$ and $RI \subset I$. An ideal in R is an example of a $R-$hypermodule.

(4) If R and S are hyperrings with $R \subset S$, then S is an example of a $R-$hypermodule.

(5) If R and S are hyperrings, and $f : R \to S$ is a strong homomorphism, that is such that $f(r + s) = f(r) + f(s)$, then any $S-$module A is also given a structure of a $R-$module by $r \cdot a = f(r) \cdot a$.

2 Quotients

Let R be a hyperring and A and B two $R-$hypermodules with $A \supset B$. For $a \in A$ define $\overline{a} = a + B = \cup\{a + b : b \in B\}$, and let $A/B = \{\overline{a} : a \in A\}$. We define addition in A/B by

$$\overline{a} \in \overline{b} + \overline{c} \text{ if and only if } a \in b + c,$$

$- : A/B \to A/B$ is defined by $-\overline{a} = \overline{-a}$, the zero element of A/B is $\overline{0}$, and the multiplication is $r\overline{a} = \overline{ra}$.

Proposition 1. *$(A/B, +, -, \overline{0}, \cdot)$ is an $R-$hypermodule.*

Proof. It suffices to check that the addition is well-defined, the rest of the axioms follows easily. This is a bit tricky, so we shall check it in detail. Say $\overline{x} = \overline{x'}$ and $\overline{y} = \overline{y'}$, and we want to check that $\overline{x} + \overline{y} = \overline{x'} + \overline{y'}$. Fix $\overline{a} \in \overline{x} + \overline{y}$. Then, for arbitrary $b \in B$:

$$a \in x + y \subset x + b + y - b,$$

and it follows that $a \in u + v$, with $u \in x + b$ and $v \in y - b$. Since $x + b \subset x' + B$, we see that $u \in x' + b'$, for some $b' \in B$. Similarly, $v \in y' + b''$, for some $b'' \in B$. Consequently $a \in x' + b' + y' + b''$, and hence $a \in a' + b'''$ with $a' \in x' + y'$ and $b''' \in b' + b''$. In particular $a' \in a - b'''$ and $\overline{a'} \in \overline{x'} + \overline{y'}$, and it remains to show that $\overline{a} = \overline{a'}$. This is clear: for arbitrary $b'''' \in B$, $a + b'''' \subset a' + b''' + b'''' \subset a' + B$, and the other inclusion is similar. The rest of the argument follows by symmetry. $\square$

Proposition 2. *For two $R-$hypermodules A and B with $A \supset B$ we define the relation*

$$x \sim y \text{ if and only if } \exists a \in A(x \in \overline{a} \wedge y \in \overline{a}) \text{ if and only if } x - y \subset B.$$

This relation is an equivalence whose set of classes coincides with A/B.

Proof. The only nontrivial parts are checking that $\sim$ is transitive and that both definitions of $\sim$ agree. The argument resembles the one used towards the end of the Proposition 1 and hence will be omitted. □

3 Morphisms

As we deal with multivalued addition here, it seems to make more sense to define morphisms as relations rather than functions – we shall, however, stick to the "functional" notation here and think of relations as of functions whose values are sets. More precisely, let R be a hyperring and A and B two $R-$hypermodules. A **morphism** $A \xrightarrow{f} B$ between A and B is defined as a function $f : A \to 2^B$ such that

(M1) $f(a + b) = f(a) + f(b)$,

(M2) $f(ra) = rf(a)$,

(M3) $f(-a) = -f(a)$,

(M4) $f(0) = \{0\}$.

We shall denote the category of $R-$hypermodules by $R - \mathcal{HM}od$ from now on. Also, for purely aesthetical reasons, we shall denote morphisms of $R-$hypermorphisms by $A \overset{f}{\nrightarrow} B$ to emphasise that f is, in fact, a function with values in 2^B.

Remarks: (1) Note that for morphisms $A \overset{f}{\nrightarrow} B$ and $B \overset{g}{\nrightarrow} C$ the composition $A \overset{g \circ f}{\nrightarrow} C$ is defined as follows:

$$g \circ f(a) = \{c \in C : \exists b \in B[(b \in f(a)) \wedge c \in g(b)]\}.$$

(2) One of the advantages of defining morphisms as multi-valued rather than single-valued functions is that this notion enables us to define algebraic

operations on morphisms: for $A \overset{f,g}{\nrightarrow} B$, the morphisms f, g give rise to $A \overset{f+g}{\nrightarrow} B$ defined as

$$(f+g)(a) = f(a) + g(a).$$

One checks that $f+g$ is, indeed, a morphism.

(3) As a consequence of **(M1)** we get that, for a morphism $A \overset{f}{\nrightarrow} B$, $\operatorname{im} f = \{b \in B : \exists a \in A[b \in f(a)]\}$ is a hypermodule with $\operatorname{im} f \subset B$.

We need to have a clear understanding of what monomorphisms and epimorphisms are. For that purpose we shall introduce the notions of weak and strong injections and surjections. Let $A \overset{f}{\nrightarrow} B$ be a morphism of $R-$hypermodules. We say that f is **weakly injective** if

$$\forall a, b \in A[(f(a) \cap f(b) \neq \emptyset) \Rightarrow (a = b)].$$

We say that f is **strongly injective** if

$$\forall a, b \in A[(f(a) = f(b)) \Rightarrow (a = b)].$$

Remarks: (1) Clearly weakly injective morphisms are also strongly injective. ,,Strongly injective" means nothing but ,,injective as a function with values in 2^B".

(2) If a morphism $A \overset{f}{\nrightarrow} B$ is weakly injective, it gives a rise to the following equivalence relation on the set B:

$$x \sim y \text{ if and only if } \exists a \in A(x \in f(a) \wedge y \in f(a)).$$

Indeed, it suffices to check that the relation is well-defined. This is the case, for if, for some $a, b \in A$, $f(a) \cap f(b) \neq \emptyset$, then $a = b$ and, consequently, $f(a) = f(b)$.

Proposition 3. *In the category $R - \mathcal{HM}od$ if $B \overset{f}{\nrightarrow} C$ is a monomorphism then it is strongly injective.*

Proof. Suppose that $B \overset{f}{\nrightarrow} C$ is a monomorphism. Fix $b_1, b_2 \in B$ and let $f(b_1) = f(b_2)$. Define mappings $\widehat{b_1}, \widehat{b_2} : R \to 2^B$ by $\widehat{b_1}(r) = \{rb_1\}$ and $\widehat{b_2}(r) = \{rb_2\}$ (here R is viewed as a $R-$hypermodule). By **(HM9)** $\widehat{b_1}$ and $\widehat{b_2}$ are well-defined morphisms of $R-$hypermodules. Moreover, $f \circ \widehat{b_1}(r) = f(rb_1) = rf(b_1) = rf(b_2) = f(rb_2) = f \circ \widehat{b_2}(r)$, and thus $\widehat{b_1} = \widehat{b_2}$. In particular $\widehat{b_1}(1) = \widehat{b_2}(1)$, that is $b_1 = b_2$. □

Proposition 4. *In the category $R-\mathcal{HM}od$ if $B \overset{f}{\nrightarrow} C$ is weakly injective, then it is a monomorphism.*

Proof. Say $B \overset{f}{\nrightarrow} C$ is weakly injective, and let $A \overset{g,h}{\nrightarrow} B$ be two morphisms such that $f \circ h = f \circ g$. It suffices to show that $h(a) = g(a)$, for $a \in A$. Indeed, fix an $a \in A$ and let $b \in h(a)$. Then $f(b) \subset f \circ h(a) = f \circ g(a)$, and therefore, for some $b' \in g(a)$, $f(b) \cap f(b') \neq \emptyset$. Thus $b = b'$ and hence $b \in g(c)$. The other inclusion is proved analogously. □

Example: We shall provide an example of a strongly injective morphism that is not weakly injective. Consider the linearly ordered set $[0, +\infty)$ equipped with the usual multiplication and a multivalued addition defined as follows:

$$a + b = \begin{cases} \max\{a, b\}, & \text{if } a \neq b \\ [0, a], & \text{if } a = b. \end{cases}$$

The opposite element function$-: [0, +\infty) \to [0, +\infty)$ is given by $-a = a$. The structure $([0, +\infty), +, \cdot, -, 0, 1)$ is a special case of a hyperfield defined by a linear ordering studied, for example, in [6, 4.7] so, in particular, it is a hyperring and a hypermodule over itself. Define the map $[0, +\infty) \overset{f}{\nrightarrow} [0, +\infty)$ by $f(a) = [0, a]$. One checks that f is a morphism, for if $a \neq b$, then

$$f(a + b) = [0, \max\{a, b\}],$$

and

$$f(a) + f(b) = [0, a] + [0, b] = [0, \max\{a, b\}],$$

whereas for $a = b$:

$$f(a + b) = [0, a]$$

and

$$f(a) + f(b) = [0, a] + [0, a] = [0, a].$$

Clearly f is strongly injective, but not weakly injective.

Similarly, we introduce the notions of weak and strong surjections. A morphism $A \overset{f}{\nrightarrow} B$ of $R-$hypermodules will be said to be **weakly surjective** if

$$\forall b \in B \exists a \in A [b \in f(a)],$$

and **strongly surjective,** if

$$\forall B' \in 2^B \exists a \in A [B' = f(a)].$$

Remarks: (1) Clearly strongly surjective morphisms are also weakly surjective. ,,Strongly surjective" means nothing but ,,surjective as a function with values in 2^B".

(2) There are obvious examples of weakly surjective morphisms that are not strongly surjective, for example the identity map on any hypermodule is weakly surjective, but not strongly surjective.

Proposition 5. *In the category $R - \mathcal{HM}od$ if $A \overset{f}{\nrightarrow} B$ is an epimorphism then it is weakly surjective.*

Proof. Suppose that f is an epimorphism. Define morphisms $B \overset{0,\pi}{\nrightarrow} B/\operatorname{im} f$ by $0(b) = \{\overline{0}\}$ and $\pi(b) = \{\overline{b}\}$. Then:

$$\pi \circ f = \{\overline{0}\} = 0 \circ f,$$

so that $\pi = 0$, that is f is a weak surjection. □

Proposition 6. *In the category $R - \mathcal{HM}od$ if $A \overset{f}{\nrightarrow} B$ is strongly surjective, then it is an epimorphism.*

Proof. Assume that $A \overset{f}{\nrightarrow} B$ is strongly surjective and that $B \overset{g,h}{\nrightarrow} C$ are morphisms such that $g \circ f = h \circ f$. For a fixed $b \in B$ let $a \in A$ be such that $f(a) = \{b\}$. Then $g(b) = g(f(a)) = g \circ f(a) = h \circ f(a) = h(f(a)) = h(b)$. □

Remark: It follows from Propositions 3 and 5 that an isomorphism of $R-$hypermodules is strongly injective and weakly surjective.

Proposition 7. *In the category $R - \mathcal{HM}od$ a morphism $A \overset{f}{\nrightarrow} B$ is an isomorphism if and only if it is a single-valued bijective morphism.*

Proof. Firstly, assume that $A \overset{f}{\nrightarrow} B$ is an isomorphism and suppose that f is not single-valued, that is for some $a \in A$ there exist $b_1, b_2 \in f(a)$ with $b_1 \neq b_2$. Since f is an isomorphism, there exists a morphism $B \overset{g}{\nrightarrow} A$ such that $f \circ g = id_B$ and $g \circ f = id_A$. In particular, g is an isomorphism as well, and thus it is strongly injective. Moreover, $g(b_1) = \{a\}$ and $g(b_2) = \{a\}$, so that $b_1 = b_2$ – a contradiction. Hence f is single-valued, strongly injective and weakly surjective, and thus bijective as a function.

Next, suppose that $A \overset{f}{\nrightarrow} B$ is a single-valued bijective morphism. Define the map $B \overset{g}{\nrightarrow} A$ by

$$g(b) = \{a\} \text{ if and only if } f(a) = \{b\}.$$

Clearly $f \circ g = id_B$ and $g \circ f = id_A$, and to check that g is a morphism fix $b_1, b_2 \in B$ and let $a_1, a_2 \in A$ satisfy $f(a_1) = b_1$ and $f(a_2) = b_2$. Observe that:

$$\begin{aligned} a \in g(b_1 + b_2) &\Leftrightarrow a = g(b) \wedge b \in b_1 + b_2 \Leftrightarrow b = f(a) \wedge b \in b_1 + b_2 \\ &\Leftrightarrow f(a) \in f(a_1) + f(a_2) = f(a_1 + a_2) \Leftrightarrow a \in a_2 + a_2 \\ &\Leftrightarrow a \in g(b_1) + g(b_2). \end{aligned}$$

Thus $g(b_1 + b_2) = g(b_1) + g(b_2)$. □

4 Initial and terminal objects

Proposition 8. *For a fixed hyperring R the zero $R-$hypermodule $\{0\}$ is both the initial and the terminal object in the category $R - \mathcal{HM}od$. In particular, $R - \mathcal{HM}od$ is equipped with the zero object.*

Remark: Note that the uniqueness of the morphism $\{0\} \nrightarrow A$ is guaranteed by the axiom **(M4)**.

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