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The pp conjecture for the space of orderings of the field $\mathbb{R}(x, y)$

Saskatoon, October 5, 2006

K - formally real field ($\mathbb{R}$)

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Th.: If K has two square classes, then

$$(a_1, \ldots, a_n) \cong (b_1, \ldots, b_m) \text{ iff.}$$
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A concept of a 'reduced' isometry

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$b \in D(a_1, a_2)$ iff. $\forall P \in X_K[(a_1(P) = b(P)) \vee (a_2(P) = b(P))]$.

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$Y = \bigcap_{a \in S} U(a)$

$G_K|_Y$ = group of all restrictions $a|_Y$, $a \in G_K$.

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$\Psi(\underline{a})$ - pp formula, $\underline{a} = (a_1, \ldots, a_k)$:

$$\exists \underline{t} \bigwedge_{j=1}^{m} p_j(\underline{t}, \underline{a}) \in D(1, q_j(\underline{t}, \underline{a})),$$

$\underline{t} = (t_1, \ldots, t_n)$, $p_j(\underline{t}, \underline{a})$, $q_j(\underline{t}, \underline{a})$ - products of $\pm$ some of the t_i's and some of the a_l's

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$$X_n = U(x^2 + y^2 - 1) \cap U(1 + \tfrac{1}{n} - x^2 - y^2),\ G_n = G_{\mathbb{R}(x,y)}|_{X_n}$$

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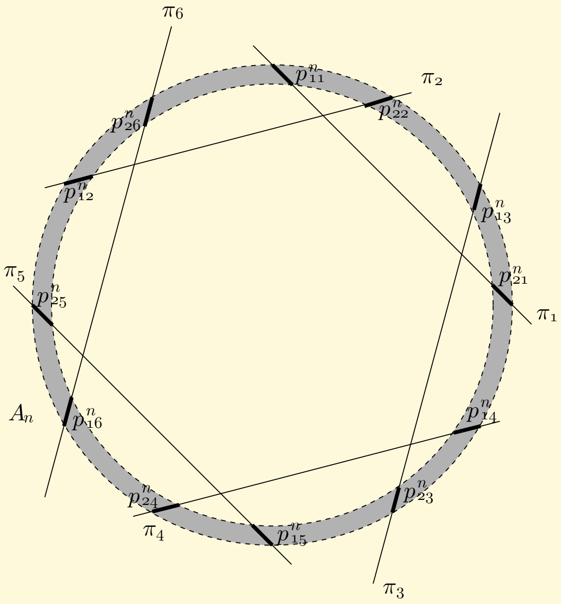
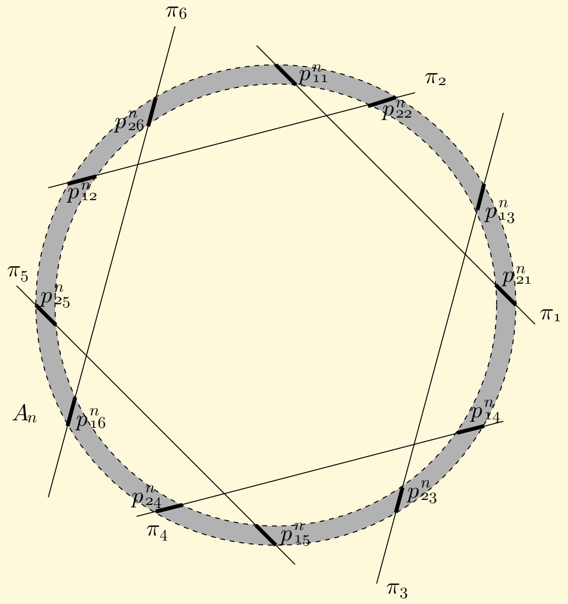
$$A_n = \{(a,b) \in \mathbb{R}^2 : 1 < a^2 + b^2 < 1 + \tfrac{1}{n}\}$$


π_6
p_{11}^n
π_2
p_{26}^n
p_{22}^n
p_{12}^n
p_{13}^n
π_5
p_{21}^n
p_{25}^n
π_1
A_n
p_{16}^n
p_{14}^n
p_{24}^n
p_{23}^n
π_4
p_{15}^n
π_3



π_6
p^n_{11}
π_2
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p^n_{26}
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π_5
p^n_{21}
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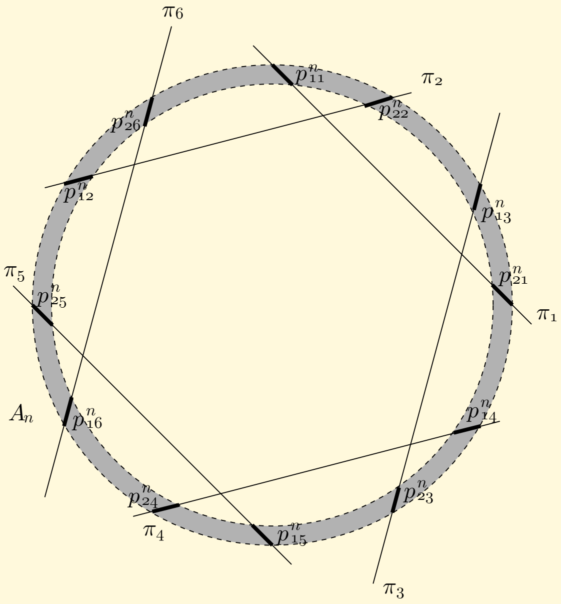
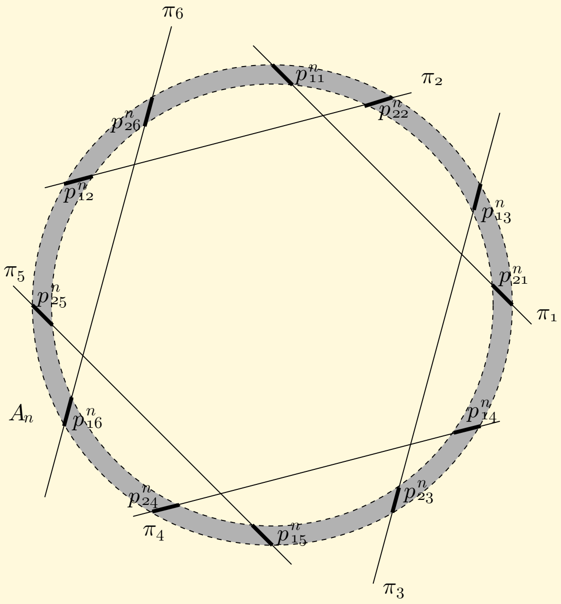
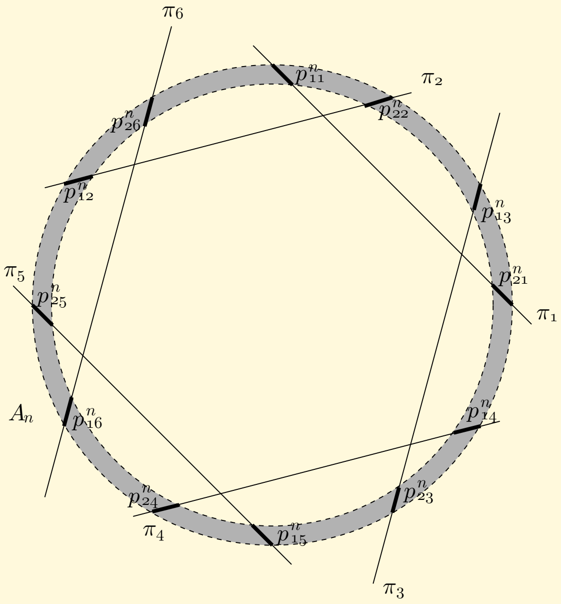
$$a_1 = \pi_1\pi_6,$$



$$a_1 = \pi_1\pi_6, \quad a_2 = \pi_1\pi_4,$$



$$a_1 = \pi_1\pi_6, \quad a_2 = \pi_1\pi_4, \quad d = -\pi_1\pi_2\pi_3\pi_5$$



	a_1	a_2	d
$A_n^{11,22}$			
$A_n^{22,13}$			
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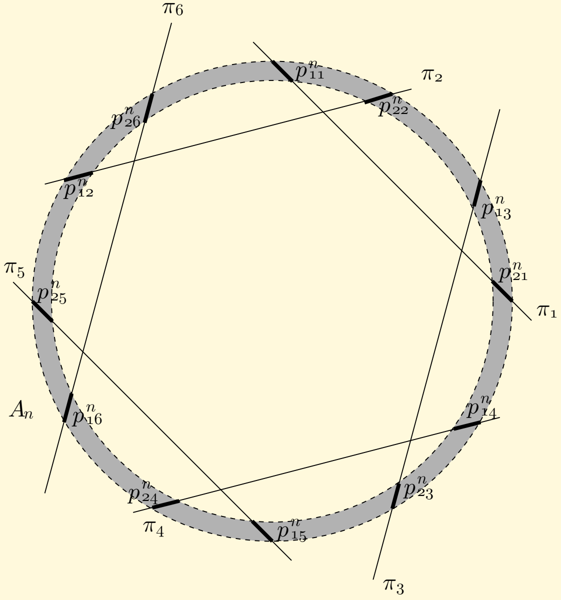
π_1: -



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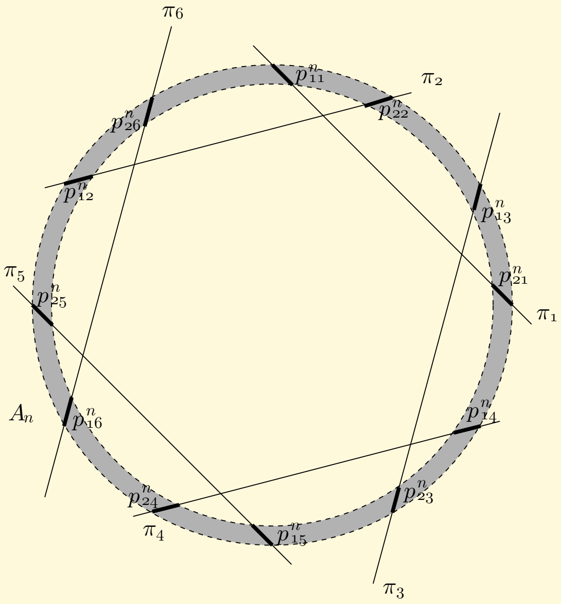
π_1: -

π_2: -



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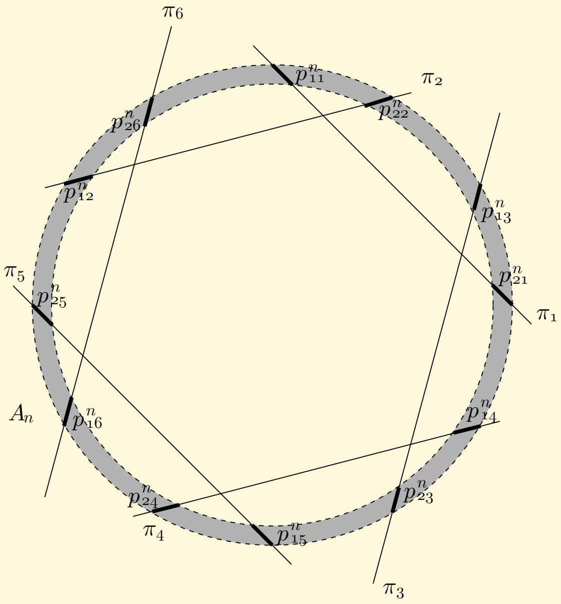
π_1: -

π_2: -

π_3: +



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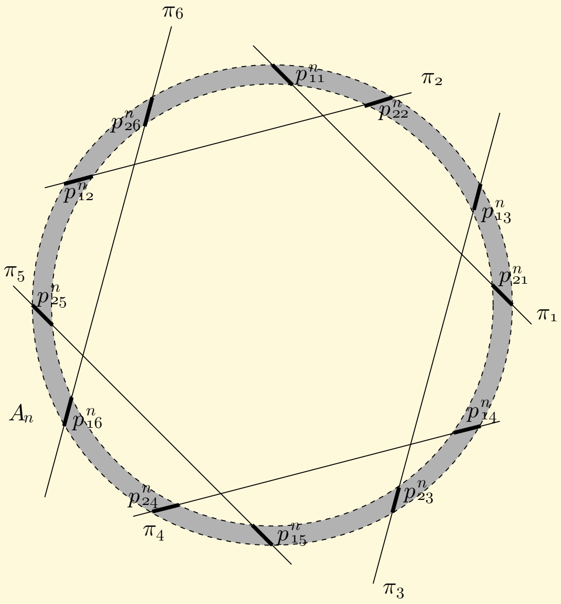
π_1: - π_4: +

π_2: -

π_3: +



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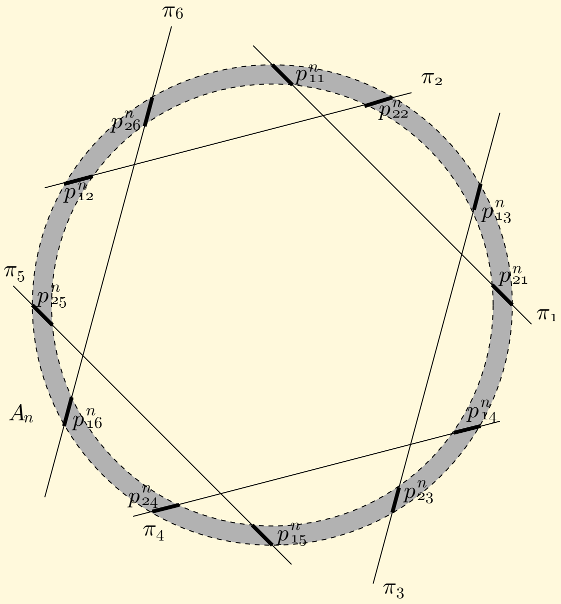

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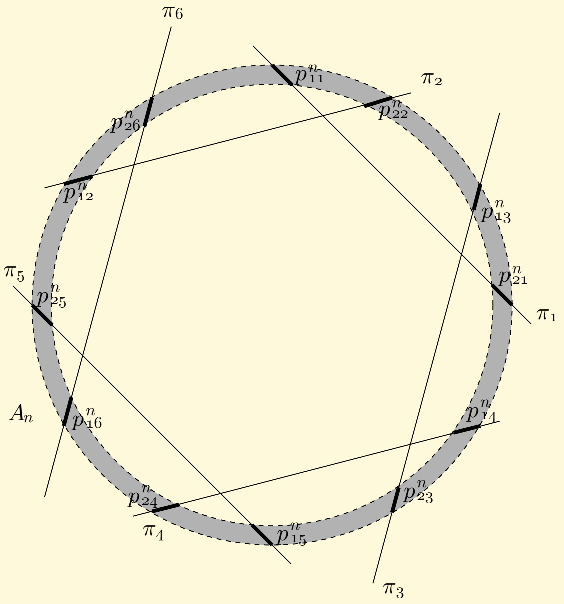
π_3: $+$ $\quad$ π_6: $+$



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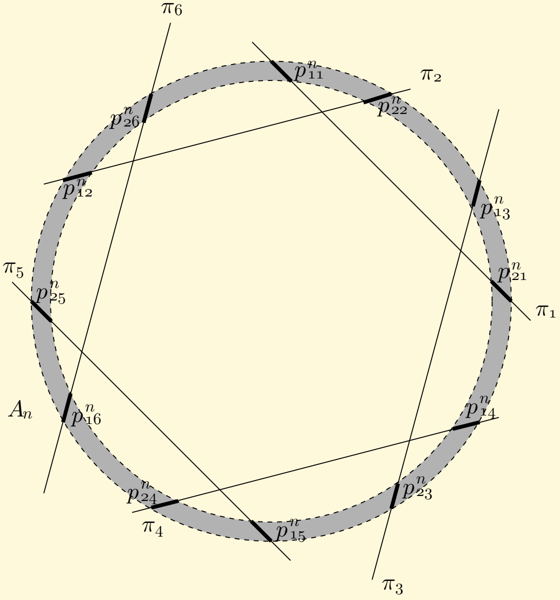

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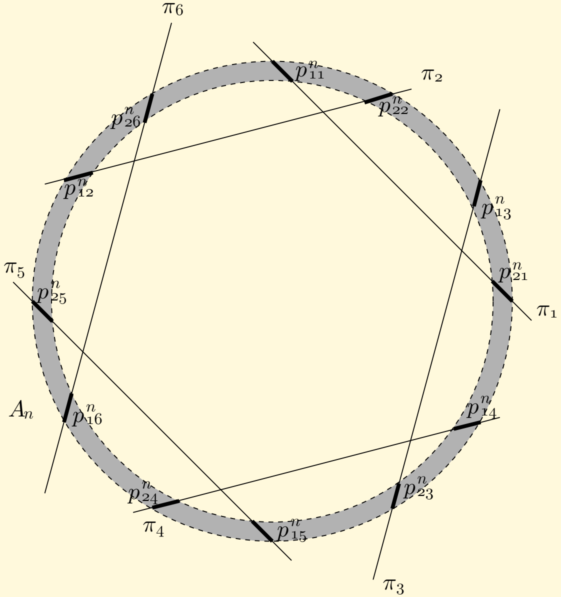

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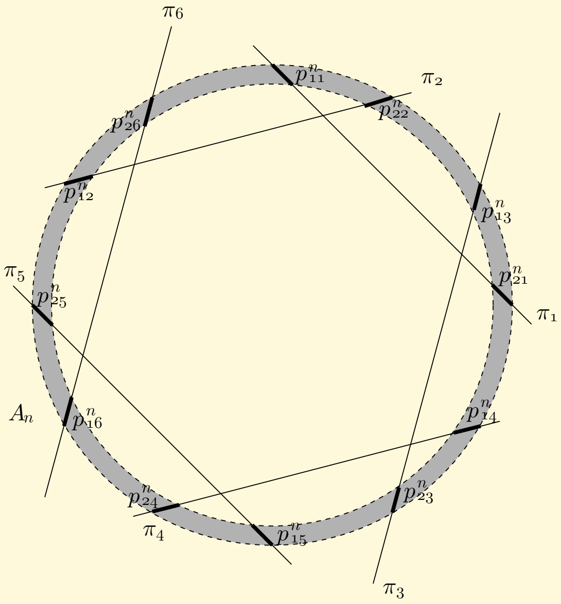
$A_n^{13,21}$			
$A_n^{21,14}$			
$A_n^{14,23}$			
$A_n^{23,15}$			
$A_n^{15,24}$			
$A_n^{24,16}$			
$A_n^{16,25}$			
$A_n^{25,12}$			
$A_n^{12,26}$			
$A_n^{26,11}$			

$$a_1 = \pi_1\pi_6, \quad a_2 = \pi_1\pi_4, \quad d = -\pi_1\pi_2\pi_3\pi_5$$

π_1: - π_4: +

π_2: - π_5: +

π_3: + π_6: +



	a_1	a_2	d
$A_n^{11,22}$	$-$	$-$	$-$
$A_n^{22,13}$			
$A_n^{13,21}$			
$A_n^{21,14}$			
$A_n^{14,23}$			
$A_n^{23,15}$			
$A_n^{15,24}$			
$A_n^{24,16}$			
$A_n^{16,25}$			
$A_n^{25,12}$			
$A_n^{12,26}$			
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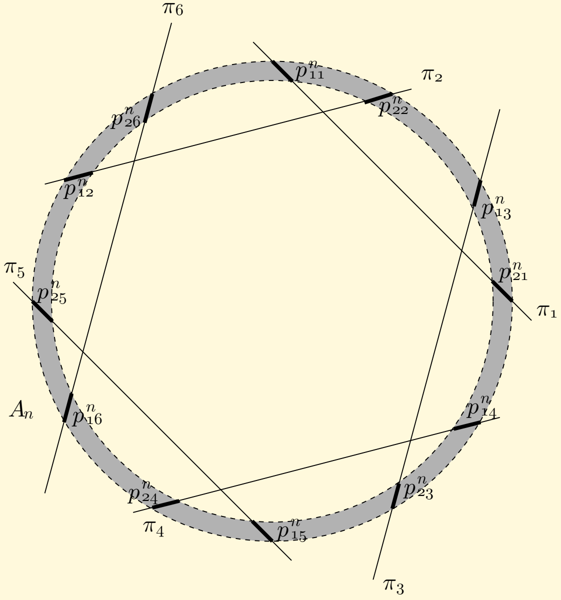
π_1: - π_4: $+$

π_2: - π_5: $+$

π_3: $+$ π_6: $+$



	a_1	a_2	d
$A_n^{11,22}$	$-$	$-$	$-$
$A_n^{22,13}$	$-$	$-$	$+$
$A_n^{13,21}$	$-$	$-$	$-$
$A_n^{21,14}$	$+$	$+$	$+$
$A_n^{14,23}$	$+$	$-$	$+$
$A_n^{23,15}$	$+$	$-$	$-$
$A_n^{15,24}$	$+$	$-$	$+$
$A_n^{24,16}$	$+$	$+$	$+$
$A_n^{16,25}$	$-$	$+$	$+$
$A_n^{25,12}$	$-$	$+$	$-$
$A_n^{12,26}$	$-$	$+$	$+$
$A_n^{26,11}$	$+$	$+$	$+$

$$a_1 = \pi_1\pi_6, \quad a_2 = \pi_1\pi_4, \quad d = -\pi_1\pi_2\pi_3\pi_5$$

$$P(a_1, a_2, d) = \exists t_1 \exists t_2 (t_1 \in D(1, a_1) \wedge t_2 \in D(1, a_2) \wedge d t_1 t_2 \in D(1, a_1 a_2))$$

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Claim 1: $P(a_1, a_2, d)$ fails to hold in (X, G)

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Claim 2: $P(a_1, a_2, d)$ holds true on every finite subspace

Lemma: $P(a_1, a_2, d)$ can be written in the form

$$d \in D(1, a_1) D(1, a_2) D(1, a_1 a_2)$$

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Th. Let (Y, H) be a subspace, let $d \in D((1, a_1) \otimes (1, a_2))$. Then $d \in D(1, a_1) D(1, a_2) D(1, a_1 a_2)$ in (Y, H) if and only if for every connected component (Y_0, H_0) of (Y, H) which is not a fan, if $a_1, a_2 \in \overline{H}$ (where $(\overline{Y}, \overline{H})$ denotes the residue space of (Y_0, H_0)), neither a_1, a_2 nor $a_1 a_2$ is equal to -1, $(1, a_1) \otimes (1, a_2)$ is isotropic over (Y_0, H_0), then $d \in \overline{H}$.

$$P(a_1, a_2, d) = \exists t_1 \exists t_2 (t_1 \in D(1, a_1) \wedge t_2 \in D(1, a_2) \wedge d t_1 t_2 \in D(1, a_1 a_2))$$

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M. Marshall, *Local-global properties of positive primitive formulas in the theory of spaces of orderings*, to appear

$$d \notin D((1, a_1) \otimes (1, a_2))$$

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Tarski Transfer Principle: there is a point $(a, b) \in A_n$ such that

$$a_1(a, b) > 0, a_2(a, b) > 0, d(a, b) < 0$$

$$d \in D((1, a_1) \otimes (1, a_2))$$

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There exists a connected component (Y_0, H_0) of (Y, H), which is not a fan, such that $a_1, a_2 \in \overline{H}$, neither a_1, a_2 nor a_1a_2 is equal to -1, $(1, a_1) \times (1, a_2)$ is isotropic over (Y_0, H_0) and $d \notin \overline{H}$

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$$(Y, H) = (Y_0, H_0)$$

$$a_1, a_2, a_1 a_2 \neq -1 \Rightarrow$$

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there are elements of $\overline{Y}$ making a_1, a_2 and a_1a_2 positive

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$(1, a_1) \otimes (1, a_2)$ is isotropic $\Rightarrow$

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there is no element of $\overline{Y}$ making both a_1 and a_2 postive

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let $\sigma_1, \sigma_2, \sigma_3 \in \overline{Y}$ be such that a_1, a_2 and a_1a_2 have the following signs:

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there is no element of $\overline{Y}$ making both a_1 and a_2 postive

let $\sigma_1, \sigma_2, \sigma_3 \in \overline{Y}$ be such that a_1, a_2 and a_1a_2 have the following signs:

	σ_1	σ_2	σ_3
a_1	+	-	-
a_2	-	+	-
a_1a_2	-	-	+

consider the subspace which is not a fan and for which $\{\sigma_1, \sigma_2, \sigma_3\}$ is a minimal generating set

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let (Y_1, H_1) be its group extension by d

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let (Y_1, H_1) be its group extension by d

(Y_1, H_1) consists of 6 orderings $\sigma_1^+, \sigma_2^+, \sigma_3^+, \sigma_1^-, \sigma_2^-, \sigma_3^-$, with respect to which the signs of a_1, a_2, a_1a_2, d are as follows:

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	σ_1^+	σ_2^+	σ_3^+	σ_1^-	σ_2^-	σ_3^-
a_1	+	-	-	+	-	-
a_2	-	+	-	-	+	-
a_1a_2	-	-	+	-	-	+
d	+	+	+	-	-	-

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	σ_1^+	σ_2^+	σ_3^+	σ_1^-	σ_2^-	σ_3^-
a_1	+	-	-	+	-	-
a_2	-	+	-	-	+	-
a_1a_2	-	-	+	-	-	+
d	+	+	+	-	-	-

$(Y, H) = (Y_1, H_1)$.

$$
\begin{aligned}
V^{11,22} &= U(-\pi_1) \cap U(-\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{22,13} &= U(-\pi_1) \cap U(\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{13,21} &= U(-\pi_1) \cap U(\pi_2) \cap U(-\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{21,14} &= U(\pi_1) \cap U(\pi_2) \cap U(-\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{14,23} &= U(\pi_1) \cap U(\pi_2) \cap U(-\pi_3) \cap U(-\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{23,15} &= U(\pi_1) \cap U(\pi_2) \cap U(\pi_3) \cap U(-\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{15,24} &= U(\pi_1) \cap U(\pi_2) \cap U(\pi_3) \cap U(-\pi_4) \cap U(-\pi_5) \cap U(\pi_6) \\
V^{24,16} &= U(\pi_1) \cap U(\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(-\pi_5) \cap U(\pi_6) \\
V^{16,25} &= U(\pi_1) \cap U(\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(-\pi_5) \cap U(-\pi_6) \\
V^{25,12} &= U(\pi_1) \cap U(\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(-\pi_6) \\
V^{12,26} &= U(\pi_1) \cap U(-\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(-\pi_6) \\
V^{26,11} &= U(\pi_1) \cap U(-\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(\pi_6).
\end{aligned}
$$

Tarski Transfer Principle: $V^{i_1 j_1, i_2 j_2}$ form a partition of (X, G)

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signs of a_1, a_2 and d on the $V^{i_1 j_1, i_2 j_2}$ are exactly the same as on the sector $A_n^{i_1 j_1, i_2 j_2}$, for respective i_1, i_2, j_1, j_2:

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	$V^{11,22}$	$V^{22,13}$	$V^{13,21}$	$V^{21,14}$	$V^{14,23}$	$V^{23,15}$	$V^{15,24}$	$V^{24,16}$	$V^{16,25}$	$V^{25,12}$	$V^{12,26}$	$V^{26,11}$
a_1	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$
a_2	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$
d	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$

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	$V^{11,22}$	$V^{22,13}$	$V^{13,21}$	$V^{21,14}$	$V^{14,23}$	$V^{23,15}$	$V^{15,24}$	$V^{24,16}$	$V^{16,25}$	$V^{25,12}$	$V^{12,26}$	$V^{26,11}$
a_1	−	−	−	+	+	+	+	+	−	−	−	+
a_2	−	−	−	+	−	−	−	+	+	+	+	+
d	−	+	−	+	+	−	+	+	+	−	+	+

$\sigma_1^- \in V^{23,15}$,

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	$V^{11,22}$	$V^{22,13}$	$V^{13,21}$	$V^{21,14}$	$V^{14,23}$	$V^{23,15}$	$V^{15,24}$	$V^{24,16}$	$V^{16,25}$	$V^{25,12}$	$V^{12,26}$	$V^{26,11}$
a_1	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$
a_2	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$
d	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$

$\sigma_1^- \in V^{23,15}$,

$\sigma_1^+ \in V^{14,23}$ or $\sigma_1^+ \in V^{15,24}$

Tarski Transfer Principle: $V^{i_1 j_1, i_2 j_2}$ form a partition of (X, G)

signs of a_1, a_2 and d on the $V^{i_1 j_1, i_2 j_2}$ are exactly the same as on the sector $A_n^{i_1 j_1, i_2 j_2}$, for respective i_1, i_2, j_1, j_2:

	$V^{11,22}$	$V^{22,13}$	$V^{13,21}$	$V^{21,14}$	$V^{14,23}$	$V^{23,15}$	$V^{15,24}$	$V^{24,16}$	$V^{16,25}$	$V^{25,12}$	$V^{12,26}$	$V^{26,11}$
a_1	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$
a_2	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$
d	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$

$\sigma_1^- \in V^{23,15}$,

$\sigma_1^+ \in V^{14,23}$ or $\sigma_1^+ \in V^{15,24}$

$\sigma_2^- \in V^{25,12}$

Tarski Transfer Principle: $V^{i_1 j_1, i_2 j_2}$ form a partition of (X, G)

signs of a_1, a_2 and d on the $V^{i_1 j_1, i_2 j_2}$ are exactly the same as on the sector $A_n^{i_1 j_1, i_2 j_2}$, for respective i_1, i_2, j_1, j_2:

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a_1	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$
a_2	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$
d	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$

$\sigma_1^- \in V^{23,15}$,
$\sigma_1^+ \in V^{14,23}$ or $\sigma_1^+ \in V^{15,24}$
$\sigma_2^- \in V^{25,12}$
$\sigma_2^+ \in V^{16,25}$ or $\sigma_2^+ \in V^{12,26}$

Tarski Transfer Principle: $V^{i_1 j_1, i_2 j_2}$ form a partition of (X, G)

signs of a_1, a_2 and d on the $V^{i_1 j_1, i_2 j_2}$ are exactly the same as on the sector $A_n^{i_1 j_1, i_2 j_2}$, for respective i_1, i_2, j_1, j_2:

	$V^{11,22}$	$V^{22,13}$	$V^{13,21}$	$V^{21,14}$	$V^{14,23}$	$V^{23,15}$	$V^{15,24}$	$V^{24,16}$	$V^{16,25}$	$V^{25,12}$	$V^{12,26}$	$V^{26,11}$
a_1	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$
a_2	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$
d	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$

$\sigma_1^- \in V^{23,15}$,

$\sigma_1^+ \in V^{14,23}$ or $\sigma_1^+ \in V^{15,24}$

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$\sigma_3^+ \in V^{22,13}$

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	$V^{11,22}$	$V^{22,13}$	$V^{13,21}$	$V^{21,14}$	$V^{14,23}$	$V^{23,15}$	$V^{15,24}$	$V^{24,16}$	$V^{16,25}$	$V^{25,12}$	$V^{12,26}$	$V^{26,11}$
a_1	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$
a_2	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$
d	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$

$\sigma_1^- \in V^{23,15}$,
$\sigma_1^+ \in V^{14,23}$ or $\sigma_1^+ \in V^{15,24}$
$\sigma_2^- \in V^{25,12}$
$\sigma_2^+ \in V^{16,25}$ or $\sigma_2^+ \in V^{12,26}$
$\sigma_3^+ \in V^{22,13}$
$\sigma_3^- \in V^{11,22}$ or $\sigma_3^- \in V^{13,21}$

4-element fans

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$$\{\sigma_1^+, \sigma_1^-, \sigma_2^+, \sigma_2^-\}$$

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$$\{\sigma_1^+, \sigma_1^-, \sigma_2^+, \sigma_2^-\}$$

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$\sigma_1^+ \in V^{14,23}$ and $\sigma_2^+ \in V^{12,26}$, then $\pi_1(\sigma_1^+\sigma_1^-\sigma_2^+\sigma_2^-) = -1$ - a contradiction

4-element fans

$$\{\sigma_1^+, \sigma_1^-, \sigma_2^+, \sigma_2^-\}$$

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$\sigma_1^+ \in V^{14,23}$ and $\sigma_2^+ \in V^{12,26}$, then $\pi_1(\sigma_1^+\sigma_1^-\sigma_2^+\sigma_2^-) = -1$ - a contradiction

$\sigma_1^+ \in V^{14,23}$ and $\sigma_2^+ \in V^{16,25}$, then $\pi_5(\sigma_1^+\sigma_1^-\sigma_2^+\sigma_2^-) = -1$ - a contradiction

4-element fans

$$\{\sigma_1^+, \sigma_1^-, \sigma_2^+, \sigma_2^-\}$$

$$\{\sigma_1^+, \sigma_1^-, \sigma_3^+, \sigma_3^-\}$$

$\sigma_1^+ \in V^{14,23}$ and $\sigma_2^+ \in V^{12,26}$, then $\pi_1(\sigma_1^+\sigma_1^-\sigma_2^+\sigma_2^-) = -1$ - a contradiction

$\sigma_1^+ \in V^{14,23}$ and $\sigma_2^+ \in V^{16,25}$, then $\pi_5(\sigma_1^+\sigma_1^-\sigma_2^+\sigma_2^-) = -1$ - a contradiction

$\sigma_1^+ \in V^{15,24}$

4-element fans

$$\{\sigma_1^+, \sigma_1^-, \sigma_2^+, \sigma_2^-\}$$

$$\{\sigma_1^+, \sigma_1^-, \sigma_3^+, \sigma_3^-\}$$

$\sigma_1^+ \in V^{14,23}$ and $\sigma_2^+ \in V^{12,26}$, then $\pi_1(\sigma_1^+\sigma_1^-\sigma_2^+\sigma_2^-) = -1$ - a contradiction

$\sigma_1^+ \in V^{14,23}$ and $\sigma_2^+ \in V^{16,25}$, then $\pi_5(\sigma_1^+\sigma_1^-\sigma_2^+\sigma_2^-) = -1$ - a contradiction

$\sigma_1^+ \in V^{15,24}$

$\sigma_1^+ \in V^{15,24}$ and $\sigma_3^- \in V^{13,21}$, then $\pi_3(\sigma_1^+\sigma_1^-\sigma_3^+\sigma_3^-) = -1$ - a contradiction

4-element fans

$$\{\sigma_1^+, \sigma_1^-, \sigma_2^+, \sigma_2^-\}$$

$$\{\sigma_1^+, \sigma_1^-, \sigma_3^+, \sigma_3^-\}$$

$\sigma_1^+ \in V^{14,23}$ and $\sigma_2^+ \in V^{12,26}$, then $\pi_1(\sigma_1^+\sigma_1^-\sigma_2^+\sigma_2^-) = -1$ - a contradiction

$\sigma_1^+ \in V^{14,23}$ and $\sigma_2^+ \in V^{16,25}$, then $\pi_5(\sigma_1^+\sigma_1^-\sigma_2^+\sigma_2^-) = -1$ - a contradiction

$\sigma_1^+ \in V^{15,24}$

$\sigma_1^+ \in V^{15,24}$ and $\sigma_3^- \in V^{13,21}$, then $\pi_3(\sigma_1^+\sigma_1^-\sigma_3^+\sigma_3^-) = -1$ - a contradiction

$\sigma_1^+ \in V^{15,24}$ and $\sigma_3^- \in V^{11,22}$, then $\pi_2(\sigma_1^+\sigma_1^-\sigma_3^+\sigma_3^-) = -1$ - a contradiction http://math.usask.ca/~gladki