Description of the book
"Set Theory"

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Part I. Basic set theory

1. Sets, relations, functions

   Content of the chapter: operations on sets (unions, intersection, finite products), relations, function as a relation, operations on functions (composition, image, preimage), special functions (injections, surjections, bijections, inverse functions), infinite products, Axiom of Choice (AC), equivalence relations.

   Content of commentaries: domain and range of a set, distributivity laws, components of a family of sets.

2. Orderings

   Content of the chapter: orderings (=partial orderings, i.e. relations which are reflexive, antisymmetric and transitive), linear orderings, complete orderings, morphisms of ordered sets, Tarski's Fixed Point Theorem (if every chain in X has the supremum, then every function $f\colon X \to X$ with $x \leq f(x)$ for every x has a fixed point), Hausdorff's Theorem on maximal chain, Kuratowski-Zorn Lemma, well ordered set, Theorem of Zermelo, lexicographic ordering, Dedekind cuts in an ordered set, Dedekind completions.

   Content of commentaries: extension of an ordered set to a linearly ordered set, Mac Naille completions.

3. Natural and rational numbers

   Content of the chapter: definition of the set $\mathbb{N}$ (as a well ordered unbounded set in which every bounded set has the greatest element), successor operation on $\mathbb{N}$, the theorem on definitions by induction (recursive constructions), uniqueness of $\mathbb{N}$, dependent choice, addition, multiplication and exponentiation on $\mathbb{N}$, the set $\mathbb{Q}_+$ of positive rational numbers (Archimedean Principle, Bernoulli's Inequality).

Content of commentaries: Peano arithmetic.

4. Field of the real numbers

   Content of the chapter: the set $\mathbb{R}_+$ of all positive real numbers as the Dedekind completion of the set $\mathbb{Q}_+$, algebraic operations on $\mathbb{R}_+$, the field $\mathbb{R}$ of all real numbers.

5. Equinumerosity

   Content of the chapter: equinumerous sets (the sets X and Y are equinumerous, i.e. $|X| = |Y|$, whenever there exists a bijection of X onto Y), Cantor's Theorem on the diagonal, Cantor-Bernstein Theorem, Theorem on Dichotomy, finite sets (a set is finite whenever it is equinumerous with $[0, n)$ for some $n \in \mathbb{N}$), Dedekind Pigeon Hole Principle, number of permutations of a finite set, Newton's symbol, series of positive real numbers, countable sets ($\mathbb{N} \times \mathbb{N}$ is equinumerous with $\mathbb{N}$), equinumerosity of $\mathbb{Q}$ with $\mathbb{N}$, uncountable sets, the equalities $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$ and $|\mathbb{R}| = |\mathbb{R} \setminus \mathbb{Q}|$, order characterization of $\mathbb{Q}$ (Cantor's Theorem), Theorem of Kurepa (an ordered set X is a union of countably many chains iff there exist a strictly increasing function from X into $\mathbb{Q}$), order characterization of $\mathbb{R}$, the set ${}^{\mathbb{N}}\{0, 1\}$ with the lexicographic oder.

   Content of commentaries: other bijections of $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$, a well order on the set of all finite subsets of $\mathbb{N}$, equinumerosity of an infinite set with its square, a dual form of the Cantor-Bernstein Theorem implies AC.

Part II. Axiomatic set theory

1. **Axioms.**

   Content of the chapter: axioms of the Zermelo-Fraenkel set theory (extensionality, replacement scheme, empty set, power set axiom, union axiom, infinity, foundation axiom, Axiom of Choice), theorem on comprehension, existence of the unordered pair, definition of the ordered pair, Russell's Theorem.

   Content of commentaries: remarks on other systems of axioms.

2. **Ordinal numbers.**

Content of the chapter: transitive sets, von Neumann's definition of ordinals, basic properties of ordinals, Burali-Forti Paradox, ω as the minimal inductive set, order types of well ordered sets (every well ordered set is isomorphic with an ordinal), Hartogs' Theorem, transfinite recursion, ε-induction, theorem of Zermelo (if there exists a choice function on $\mathcal{P}(X)$, then X can be well ordered), cumulative hierarchy of sets (the sets V_α), arithmetic of ordinal numbers, expansion of ordinals as sums of powers of a given α, Goodstein's Theorem.

Content of commentaries: alternative definition of ordinals, some application of Hartogs numbers (equinumerosity of every infinite X with $X \times X$ implies AC), indecomposable numbers, epsilon numbers.

3. **Cardinal numbers.**

   Content of the chapter: definition of a cardinal (as an initial ordinal), cardinal arithmetic, hierarchy of $\aleph$'s, Hessenberg's Theorem ($\kappa \cdot \kappa = \kappa$), König's Lemma, regular and singular cardinals, exponentiation of cardinals (Hausdorff formula, Tarski formula and Bukovsky Theorem)

   Content of commentaries: Kuratowski-Siepiński Lemma on refinement, epsilon numbers versus cardinal numbers, Hessenberg function, Specker's Theorem (if $|X| > 4$, then $\mathcal{P}(X)$ has no injection into $X \times X$), GCH implies AC

4. **Combinatorial properties of sets.**

   Content of the chapter: almost disjoint sets, Hausdorff gaps, Δ-lemma, Ramsey Theorem (in infinite version), König's Theorem on trees, finite version of Ramsey Theorem, Ramsey numbers (some estimations), Theorem of Schur, Theorem of Hajnal on free sets for set-valued functions, stationary sets (Fodor's Lemma, Ulam matrix), Erdós-Rado Theorem, Martin's Axiom (MA implies that $2^{\aleph_0}$ is regular, MA+$\neg$CH implies that any ordered sets with Souslin property of size less than continuum is a union of countably many centered sets).

   Content of commentaries: Ramsey numbers, Erdös-Dushnik-Miller Theorem, weakly compact cardinals and Ramsey cardinals, normal filters, some applications of Erdös-Rado Theorem in topology, Martin's Axiom and Souslin number, equivalent versions of Martin's Axiom, P-points (Booth Lemma).

Part III. Classical Constructions of Set Theory

1. **Lattices**

   Content of the chapter: notion of the lattice, homomorphisms of lattices, sublattices, characterization of distributive lattices (Theorem of Birkhoff), filters and ideals, ultrafilters (Theorem of Tarski), quotient lattices, Boolean lattices (Stone Representation Theorem).

   Content of commentaries: prime filters and prime ideals (Theorem of Nachbin), Representation Theorem for Distributive Lattices, Rudin-Keisler ordering (Theorem of Katětov), selective ultrafilters, Ramsey ultrafilters, ultraproducts (Theorem of Scott), proof of Ramsey Theorem by the use of ultraflters.

2. **Topologies.**

   Content of the chapter: notion of the topology, order topology, regular–open sets and closed–open sets, bases of topology (weight of a topological space, Cantor-Bendixon Theorem), zero-dimensional spaces (Cantor set), continuous functions, separation axioms (Hausdorff spaces, regular spaces and normal spaces, Urysohn Lemma), topological characterization of the Cantor set (Theorem of Brouwer), topological characterization of rational numbers (Theorem of Sierpinski), topological characterization of irrational numbers (Alexandroff-Urysohn Theorem), metric spaces, complete spaces (Baire Category Theorem, completion of a metric space), Urysohn Theorem (every separable metric space is isometric with a subspace of the universal metric space $\mathbb{U}$), compact spaces, Čech complete spaces, product of topological spaces (Tychonoff Theorem, Cantor cubes, Hewitt-Marczewski-Pondiczery Theorem) completely regular spaces (Čech-Stone compactification), Stone spaces, extremely disconnected spaces, characterization of extremally disconnected compact space (Gleason Theorem).

   Content of commentaries: lattice of topologies on a set, ideal of nowhere dense sets, regular open sets versus G_δ-sets, further properties of the Čech-Stone compactification, Theorem of Taĭmanov, Gleason spaces (absolutes of topological spaces), topological reformulation of Martin's Axiom.

3. **Trees**

   Content of the chapter: notion of the tree, trees versus linearly ordered sets, the set ${}^{<\alpha}X$ as a tree, the set $wo(\mathbb{Q})$ (example of Sierpinski), König's Theorem, construction of an Aronszajn tree, special

Aronszajn tree (under Martin's Axiom all Aronszajn trees are special), construction of a Suslin tree from $\diamondsuit$.

Content of commentaries: product of Suslin trees and Suslin property, generalized Aronszajn tree, Kurepa tree.

4. **Measures**

   Content of the chapter: elementary construction of the Lebesgue measure on $\mathbb{R}$ (without AC), Steinhaus Theorem (the set of all algebraic differences of two sets of positive measure contains an interval), non-measurable sets, measures on σ-fields, Borel measures, measurable cardinals, Ulam-measurable cardinals, Ulam matrix, non-measurability of Hausdorff gaps, measurability and Martin's Axiom.

   Content of commentaries: Cantor-Lebesgue function, measure on the Cantor set, density topology, Lebesgue Theorem on the density points, Luzin-Mienshoff Theorem.

5. **Boolean algebras**

   Content of the chapter: notion of Boolean algebra, different representations of Boolean algebras (as a Boolean lattice, as Boolean ring and as a field of clopen subsets of a Stone space), complete Boolean algebras, algebra of regular open sets, completion of a Boolean algebra, free Boolean algebras, independent sets (Theorem of Shelah), algebra $\mathcal{P}(\omega)/Fin$, algebra $Bor(X)/\mathcal{K}$ of Borel sets modulo ideal of first category sets, algebra $Bor(\mathbb{R})\mathcal{N}$ of Borel sets modulo sets of Lebesgue measure zero, measures on Boolean algebras.

   Content of commentaries: characterization of atomic Boolean algebras, characterization of $\mathcal{P}(\omega)/Fin$, saturation of Boolean algebras, independent sets (Balcar-Franek Theorem, Engelking-Karłowicz Theorem), Suslin algebra, algebra with a measure.

6. **Ramsey theory**

   Content of the chapter: compact semigroups (Theorem of Numakura on idempotents), the set of ultrafilters over $\mathcal{P}(\omega)$ as a semigroup, Hindman's Theorem (every partition of ω contains an element which contains an infinite subset together with all finite unions), Baumgartner's Theorem on partitions of $[\omega]^{<\aleph_0}$, theorem of Hales-Jewett on combinatorial lines, theorem of van der Waerden on arithmetic progressions.

Content of commentaries: equivalence of Hindman's Theorem and Baumgartner's Theorem, topological version of the van der Waerden's Theorem.

Part IV. Around the Axiom of Choice

1. **Equivalent versions of the Axiom of Choice**

   Content of the chapter: multiple choice and his equivalence with AC, theorem of Blass (existence of basis in vector spaces is equivalent with AC), theorem of Kelley (Tychonoff's Theorem for compact not necessary Hausdorff spaces is equivalent with AC), Theorem of Tarski (on ultrafilters in lattices).

   Content of commentaries: Dual Dychotomy of Lindenbaum, theorem on irreducible map, theorem of Hajnal on free sets.

2. **Weaker versions of the Axiom of Choice**

   Content of the chapter: Theorem on prime ideals in Boolean Algebras (BPI), equivalence of BPI with Tychonoff's Theorem for compact Hausdorff spaces, non-measurability of free ultrafilters, Dependent Choice, equivalent forms of the Dependent Choice (Rasiowa-Sikorski Lemma , Baire Category Theorem), countable Axiom of Choice, essence of countable Axiom of Choice in equivalence of Cauchy and Heine notions of continuity, Axiom of Determinacy and his consequences (all subset of the real line are Lebesgue measurable, $\aleph_1$ is a measurable cardinal).

   Content of commentaries: consequences of BPI (products of non-empty compact Hausdorff spaces are non-empty, existence of an extension of any order to a linear order), weak form of the Baire Category Theorem.

3. **Banach-Tarski Paradox**

   Content of the chapter: action of a group on a set, equidecomposable sets, paradoxical sets, free groups, Banach-Tarski paradox.

   Content of commentaries: Banach-Tarski paradox and extensions of Lebesgue measure in $\mathbb{R}^n$, Banach-Tarski paradox versus Axiom of Choice.