EXPONENTIAL ATTRACTOR FOR THE CAHN-HILLIARD-OONO EQUATION IN $\mathbb{R}^N$

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Abstract. We consider the Cahn-Hilliard-Oono equation in the whole of $\mathbb{R}^N$, $N \le 3$. We prove the existence of an exponential attractor in $H^1(\mathbb{R}^N)$, which contains a global attractor. We also estimate from above fractal dimension of the attractors.

1. Introduction

The Cahn-Hilliard equation is well known in material sciences involving phase separation processes, see e.g. [8, 52, 46, 43] and references therein. Several variations are also of broader interest; multi-component alloy models [29, 11, 36], models with viscosity or with inertial terms [28, 45, 40, 9, 31], the Cahn-Hilliard-Cook equations [3, 4, 51], and the Cahn-Hilliard-Oono model [41, 30], which in turn involves hyperbolic relaxation models [49, 50] and Navier-Stokes equations [44].

In contrast with the case of bounded domains much less references deal with the Cahn-Hilliard type models in unbounded domains; see [7, 6, 35, 5, 38, 23, 22, 14, 15, 48, 49, 50, 55, 54] in chronological order. Specifically aspects of stability theory related to analysis of the models in $\mathbb{R}^N$ are worth further research.

In this paper given $\delta > 0$ we consider in $\mathbb{R}^N$ the Cahn-Hilliard-Oono equation

$$u_t + \Delta(\Delta u + f(x,u)) + \delta u = 0,\ t > 0,\ x \in \mathbb{R}^N, \tag{1.1}$$

subject to the initial condition

$$u(0,x) = u_0(x),\ x \in \mathbb{R}^N, \tag{1.2}$$

where $u_0 \in H^1(\mathbb{R}^N)$ and $N \le 3$. Without term δu (1.1) becomes the local Cahn-Hilliard equation which can be obtained e.g. from [39, (2)] or [26, (1.17)] by taking unitary thickness parameter and considering, moreover, the mobility term therein as unitary constant. The latter simplification in the model is often exploited in the literature and our considerations remain in this vein. Specifically, we build upon the approach in [14] suitable for the non-degenerate problems, whereas the case of concentration-dependent mobilities, which may vanish (see e.g. [27]), leads to the consideration of degenerate problems.

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As noted in [42, p. 483] and [43, p. 7] the presence in (1.1) of term δu is to account for long-ranged (i.e., nonlocal) interactions. We remark in turn that recently nonlocal Cahn-Hilliard equation has been studied in [26] in the case of a degenerate mobility (see also [25] for Vlasov-Cahn-Hilliard equation), which further expanded the profound nonlocal-to-local analysis, carried out before in [20] and [21]. We do not pursue this matter here concentrating on the approach adequate for the analysis of asymptotics of (1.1)-(1.2) as we describe below.

A small linear term δu essentially influences long-time dynamics of the Cauchy problem (1.1)-(1.2). Namely, if $\delta = 0$, then under dissipativity mechanism customarily used for parabolic problems in large unbounded domains only a weak form of dissipativity is known, that is, each individual solution is attracted by the set of equilibria (see [14]). On the other hand, if $\delta > 0$ then strong dissipativity occurs (see again [14]); in particular a semigroup of global solutions to (1.1), $\{S(t)\}_{t\geq 0}$, possesses a global attractor $\mathbf{A}$ in the sense of [32], that is, $\mathbf{A}$ is compact, invariant under $\{S(t)\}_{t\geq 0}$ and such that for each bounded subset B of $H^1(\mathbb{R}^N)$

$$d(S(t)B, \mathbf{A}) = \sup_{v\in B} \inf_{w\in \mathbf{A}} \|S(t)v - w\|_{H^1(\mathbb{R}^N)} \to 0 \quad \text{as} \quad t \to \infty.$$

In [14] the existence of the global attractor has been proven for nonlinearities which included as a sample a 'logistic' case

$$f(x, u) = m(x)u - u|u|^{\rho-1} \tag{1.3}$$

where

$$\rho > 1 \text{ is arbitrarily large if } N = 1, 2 \text{ and } 1 < \rho < \rho_c := \frac{N+2}{N-2} = 5 \text{ if } N = 3, \tag{1.4}$$

$$m \in L^r_U(\mathbb{R}^N) = \{\phi \in L^r_{loc}(\mathbb{R}^N) \colon \sup_{y\in\mathbb{R}^N} \|\phi\|_{L^r(B(y,1))} < \infty\} \quad \text{for some} \quad 2 \leq r \leq \infty, \tag{1.5}$$

and m is the sum, $m = m_1 + m_2$, of a potential m_1 for which bottom spectrum of $-(\Delta + m_1(\cdot))$ in $L^2(\mathbb{R}^N)$ is positive, and m_2 such that $|m_2(\cdot)|^{\frac{\rho}{\rho-1}} \in L^s(\mathbb{R}^N)$ for a certain

$$\max\left\{1, \frac{2N}{N+2}\right\} \leq s \leq 2 \quad (s > 1 \text{ if } \ N = 2). \tag{1.6}$$

In this paper, having indicated in $H^1(\mathbb{R}^N)$ a suitable closed (although not compact) positively invariant absorbing set $\mathbf{B}$, we will exhibit that with such f there actually exists an exponential attractor $\mathbf{M}$ for $\{S(t)\}_{t\geq 0}$, a notion having its origins in [24], namely $\mathbf{M} \subset \mathbf{B}$ such that

(i) $\mathbf{A} \subset \mathbf{M}$ and $S(t)\mathbf{M} \subset \mathbf{M}$ for every $t \geq 0$,
(ii) $\mathbf{M}$ is compact in $H^1(\mathbb{R}^N)$ and has finite fractal dimension $\dim_f(\mathbf{M}) = \limsup_{\varepsilon\to 0} \log_{\frac{1}{\varepsilon}} n_\varepsilon$, where n_ε is the smallest number of $\varepsilon-$balls needed to cover $\mathbf{M}$ in the considered space,
(iii) there exists $\omega > 0$ such that for any B bounded in $H^1(\mathbb{R}^N)$

$$e^{\omega t} d(S(t)B, \mathbf{M}) \to 0 \text{ as } t \to \infty.$$

To introduce a general assumption on the nonlinear term f in (1.1) we consider as in [14] a structure condition

$$\begin{aligned} &uf(x, u) \leq C(x)u^2 + D(x)|u|, \ x \in \mathbb{R}^N, \ u \in \mathbb{R} \text{ for some} \\ &C \in L^r_U(\mathbb{R}^N),\ D \in L^s(\mathbb{R}^N) \text{ and } r, s \text{ as in (1.5) and (1.6)} \end{aligned} \tag{1.7}$$

and C also such that the linear problem

$$\begin{cases} u_t = \Delta u + C(x)u, \ t > 0, \ x \in \mathbb{R}^N, \\ u(0) = u_0 \in L^2(\mathbb{R}^N) \end{cases} \tag{1.8}$$

enjoys uniform exponential stability property, equivalently (see [14, (A.2)])

$$\int_{\mathbb{R}^N} \left(|\nabla \phi|^2 - C(x)\phi^2 \right) \geq \omega_0 \|\phi\|^2_{L^2(\mathbb{R}^N)} \text{ for } \phi \in H^1(\mathbb{R}^N) \text{ with some } \omega_0 > 0. \tag{1.9}$$

Also, we consider an additional condition

$$uf(x,u) - \tilde{\nu} \int_0^u f(x,y)dy \leq G(x)u^2 + H(x)\,|u|\,, \ x \in \mathbb{R}^N, u \in \mathbb{R}, \tag{1.10}$$

for sufficiently small $\tilde{\nu} > 0$ and some

$$G \in L^{\tilde{r}}_U(\mathbb{R}^N), \ \tilde{r} \geq 2, \quad 0 \leq H \in L^{\tilde{s}}(\mathbb{R}^N), \ \max\left\{1, \frac{2N}{N+2}\right\} \leq \tilde{s} \leq 2 \ (\tilde{s} > 1 \text{ if } N = 2) \tag{1.11}$$

such that

$$\int_{\mathbb{R}^N} \left(|\nabla \phi|^2 - G(x)\phi^2 \right) \geq \tilde{\omega}_0 \|\phi\|^2_{L^2(\mathbb{R}^N)} \text{ for } \phi \in H^1(\mathbb{R}^N) \text{ with some } \tilde{\omega}_0 > 0. \tag{1.12}$$

Example. *i) For $f(x,u) = m(x)u - u|u|^{\rho-1}$ in (1.3) considered with $m = m_1 + m_2$ and $\rho > 1$, after using Young's inequality as in* [14, (7.2)], *we see that if $m_1 = -c$ for some constant $c > 0$ and $m_2 \in L^{\frac{2\rho}{\rho-1}}(\mathbb{R}^N)$ then (1.7)-(1.9) hold with*

$$C(x) = m_1, \qquad D(x) = \frac{\rho - 1}{\rho} |m_2|^{\frac{\rho}{\rho-1}}.$$

This remains true if instead of constant $m_1 = -c$ we take (nonconstant) nonpositive $m_1 \in L^\infty(\mathbb{R}^N)$ such that the integral $\int_G m_1 dx$ is infinite for all G from the family of open sets in $\mathbb{R}^N$ containing arbitrarily large balls, as in that case (1.8) enjoys uniform exponential stability property with $C(x) = m_1(x)$ (see [1, Theorem 1.2]*). When $N = 1$ this remains true if we take (nonconstant) nonpositive $m_1 \in L^r_U(\mathbb{R}^N)$ (see* [1, Theorem 3.5]*).*
ii) For f in i) we also obtain as in [14, (8.28)] *that*

$$uf(x,u) - \tilde{\nu} \int_0^u f(x,y)dy \leq \left(1 - \frac{\tilde{\nu}}{2}\right) uf(x,u)$$

and hence we see that (1.10)-(1.12) hold with

$$G(x) = \left(1 - \frac{\tilde{\nu}}{2}\right) C(x), \qquad H(x) = \left(1 - \frac{\tilde{\nu}}{2}\right) D(x)$$

for $C(x)$ and $D(x)$ as in i) above.

Below, not restricting ourselves to a sample f in (1.3), we will assume (1.7)-(1.12) considering

$$f(x,u) = g(x) + m(x)u + f_0(x,u), \ x \in \mathbb{R}^N, \ u \in \mathbb{R}, \tag{1.13}$$

with m as in (1.5),

$$g \in L^2(\mathbb{R}^N) \tag{1.14}$$

and $f_0\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfying

$$f_0(x,0) = 0, \quad \frac{\partial f_0}{\partial u}(x,0) = 0, \ x \in \mathbb{R}^N \tag{1.15}$$

and

$$|f_0(x,u_1) - f_0(x,u_2)| \le c|u_1 - u_2|(1 + |u_1|^{\rho-1} + |u_2|^{\rho-1}), \ u_1, u_2 \in \mathbb{R}, \ x \in \mathbb{R}^N, \tag{1.16}$$

for some ρ as in (1.4).

In [14, Theorem 8.7] it was shown that if f in (1.13)-(1.16) satisfies (1.7)-(1.9) then the problem (1.1)-(1.2) is globally well-posed in $H^1(\mathbb{R}^N)$ and the associated C^0 semigroup $\{S(t)\}_{t\ge 0}$ in $H^1(\mathbb{R}^N)$ has bounded orbits of bounded sets and for each $t > 0$, $S(t)$ maps bounded sets of $H^1(\mathbb{R}^N)$ into bounded sets of $H^2(\mathbb{R}^N)$. There is also a set B_0 bounded in $H^2(\mathbb{R}^N)$, which absorbs bounded subsets of $H^1(\mathbb{R}^N)$. Then, letting for some large $t_0 > 0$

$$\mathcal{B} = \mathrm{cl}_{H^1(\mathbb{R}^N)} \bigcup_{t \ge t_0} S(t) B_0, \tag{1.17}$$

we see that $\mathcal{B} \subset \mathrm{cl}_{H^1(\mathbb{R}^N)} B_0$, $\mathcal{B}$ absorbs bounded sets of $H^1(\mathbb{R}^N)$ and is positively invariant under $\{S(t)\}_{t\ge 0}$.

In [14, Theorem 8.9] it was also shown that if, in addition, (1.10)-(1.12) hold then for each B bounded in $H^1(\mathbb{R}^N)$ and for arbitrarily chosen $\xi > 0$ there exist certain $\tau_\xi > 0$ and $R_\xi > 0$ such that

$$\sup_{u_0 \in B} \sup_{t \ge \tau_\xi} \|S(t)u_0\|_{L^2(\{|x| > R_\xi\})} < \xi. \tag{1.18}$$

Consequently, $\{S(t)\}_{t\ge 0}$ is asymptotically compact in $H^1(\mathbb{R}^N)$ and has a global attractor $\mathbf{A}$.

In the present paper our main result is the following theorem.

Theorem 1.1. *Given $N \le 3$ assume that f is as in (1.13)-(1.16) and (1.7)-(1.12) hold. Assume, in addition, that*

$$\frac{\partial f_0}{\partial u}(x,u) \to 0 \ \text{as} \ u \to 0 \ \text{uniformly for} \ x \in \mathbb{R}^N.$$

Then there exists an exponential attractor $\mathbf{M}$ for (1.1)-(1.2) in $H^1(\mathbb{R}^N)$ as in (i)-(iii) above.

With the construction of an exponential attractor $\mathbf{M}$, the results of the present paper essentially broaden information available so far for the Cahn-Hilliard-Oono problem in $\mathbb{R}^N$ ensuring also finite fractal dimensionality of the global attractor $\mathbf{A}$ which is contained in $\mathbf{M}$.

Mention should be made that in our analysis we exhibit a suitable decomposition of the flow into exponentially stable and compact parts, which is related to the quasi-stability method developed by I. Chueshov and I. Lasiecka [17, 18]. Regarding this we prove the following result, in which we use the notion of *quasi-stable dynamical system* described in Definition A.2 of the Appendix.

Theorem 1.2. *Assume conditions of Theorem 1.1 and let $\mathcal{B}$ be as in (1.17).*

Then there is $\tau > 0$ such that for t_* chosen suitably large dynamical system $(H^1(\mathbb{R}^N), S(t))$ is quasi-stable on $\mathbf{B} = \mathrm{cl}_{H^1(\mathbb{R}^N)} S(\tau)\mathcal{B}$ at time t_*.

Within the mentioned approach we obtain in particular estimates from above of fractal dimension of exponential and global attractors, in which crucial is the compactness of the embedding of the space

$$Z = \{z \in L^2(0,t_*; H^2(\mathbb{R}^N)) \colon \frac{dz}{dt} \in L^2(0,t_*; (H^2(\mathbb{R}^N))')\}$$

into weighted-type space

$$L^2(0,t_*;L^{2r'}_{\varphi}(\mathbb{R}^N)),$$

where r' is Hölder's conjugate to r given in (1.5), t_* is as in Theorem 1.2 and $\varphi(\cdot)=\frac{1}{(1+|\cdot|^2)^{\varphi_0}}$ with $\varphi_0>\frac{N}{2}$. Using this we prove the following theorem, where in (1.21)-(1.22) the bounds of dimensions of $\mathbf{A}$ and $\mathbf{M}$ are moreover expressed via [19] through the Kolmogorov $\varepsilon-$entropy, which is known to measure quantitatively the compactness of the embedding of the above space Z into $L^2(0,t_*;L^{2r'}_{\varphi}(\mathbb{R}^N))$ (see [34] and [53, page 108]).

Theorem 1.3. *Under the conditions of Theorem 1.1 for each $0<q_*<1$ there exist $t_*>0$ and $\kappa_*,\xi_*>0$ such that given any $\theta\in(0,1-q_*)$ fractal dimension of the attractor $\mathbf{A}$ satisfies*

$$\dim_f(\mathbf{A})\le \log_{\frac{1}{q_*+\theta}}\mathfrak{m}_{\frac{2\kappa_*}{\theta}} \tag{1.19}$$

and $\mathbf{A}$ is contained in an exponential attractor $\mathbf{M}$ having fractal dimension estimated by

$$\dim_f(\mathbf{M})\le 1+\log_{\frac{1}{q_*+\theta}}\mathfrak{m}_{\frac{2\kappa_*}{\theta}}, \tag{1.20}$$

where $\mathfrak{m}_{\frac{2\kappa_*}{\theta}}$ is the maximal number of points z_j in a closed ball in Z of radius $\frac{2\kappa_*}{\theta}$ around zero with the property that $\xi_*\|z_j-z_l\|_{L^2(0,t_*;L^{2r'}_{\varphi}(\mathbb{R}^N))}>1$ whenever $j\neq l$.

If $0<q_*<\frac12$ then fractal dimension of the attractor $\mathbf{A}$ also satisfies for any $\theta\in(0,\frac12-q_*)$

$$\dim_f(\mathbf{A})\le \log_{\frac{1}{2(q_*+\theta)}} N_{\frac{\theta}{\kappa_*\xi_*}} \tag{1.21}$$

and $\mathbf{A}$ is contained in an exponential attractor $\mathbf{M}$ having fractal dimension estimated by

$$\dim_f(\mathbf{M})\le 1+\log_{\frac{1}{2(q_*+\theta)}} N_{\frac{\theta}{\kappa_*\xi_*}}, \tag{1.22}$$

where $N_{\frac{\theta}{\kappa_*\xi_*}}$ is the minimal number of balls in $L^2(0,t_*;L^{2r'}_{\varphi}(\mathbb{R}^N))$ of radius $\frac{\theta}{\kappa_*\xi_*}$ needed to cover the unit ball in Z.

In Section 2 we derive the estimates of the difference of the solutions to (1.1)-(1.2) and prove Theorem 1.2. In Section 3 we prove Theorems 1.1 and 1.3. In the Appendix we include for reader's convenience some tools used in our approach.

2. Quasi-stability of the semigroup

We use the scale of Bessel potential spaces $H^{\sigma}_2(\mathbb{R}^N)$, $\sigma\in\mathbb{R}$, defined as in [53] and denoted here by $H^{\sigma}(\mathbb{R}^N)$.

We also recall that Δ in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$ is a selfadjoint operator, which has spectrum $\sigma(\Delta)=(-\infty,0]$ and generates a C^0 analytic semigroup of contractions $\{e^{\Delta t}\}_{t\ge0}$. Given any $\alpha,\gamma>0$ we have then defined $(-\Delta+\gamma)^{-\alpha}$ in $L^2(\mathbb{R}^N)$ as in [33, Definition 1.4.1], which operators are of the class $\mathcal{L}(L^2(\mathbb{R}^N))$ and are symmetric. Moreover, we have for $\gamma>0$

$$\|(-\Delta+\gamma)^{-\frac12}\|_{\mathcal{L}(L^2(\mathbb{R}^N))}=\|\frac{1}{\Gamma(\frac12)}\int_0^{\infty}s^{-\frac12}e^{-\gamma s}e^{\Delta s}ds\|_{\mathcal{L}(L^2(\mathbb{R}^N))}\le\gamma^{-\frac12}. \tag{2.1}$$

We assume the conditions of Theorem 1.1 guaranteeing, due to the results in [14] summarized in the Introduction, that (1.1)-(1.2) governs a C^0 semigroup $\{S(t)\}_{t\ge0}$ in $H^1(\mathbb{R}^N)$ with a global attractor and with a positively invariant absorbing set $\mathcal{B}$ as in (1.17).

To derive the quasi-stability estimates the following two remarks will be useful.

Remark 2.1. Since $H^2(\mathbb{R}^N)$ is a reflexive Banach space, $\mathcal{B}$ in (1.17) is bounded in $H^2(\mathbb{R}^N)$.

Remark 2.2. Assume conditions of Theorem 1.1.
I) Then from [10, Lemmas 13.3, 13.4] we have

$$m(x) - C(x) \le \frac{\omega_0}{4} + \alpha_0 D(x),\ x \in \mathbb{R}^N, \tag{2.2}$$

$$|f_0(x,u) - f_0(x,v)| \le \Big(\frac{\omega_0}{8} + c_*(|u|^{\rho-1} + |v|^{\rho-1})\Big)|u-v|,\ u,v \in \mathbb{R}^N,\ x \in \mathbb{R}^N \tag{2.3}$$

for some $\alpha_0, c_* > 0$.
II) Since $H^2(\mathbb{R}^N) \subset BUC(\mathbb{R}^N)$ and $\mathcal{B}$ is bounded in $H^2(\mathbb{R}^N)$ (see Remark 2.1), values of elements of $\mathcal{B}$ are within the range of a certain real line interval $[-C_{\mathcal{B}}, C_{\mathcal{B}}]$. Since $\mathcal{B}$ is positively invariant, denoting

$$u = S(\cdot)u_0, \quad v = S(\cdot)v_0 \ \text{ for } \ u_0, v_0 \in \mathcal{B},$$

we conclude that u and v never leave $\mathcal{B}$. As a consequence, values of u and v are within the range of $[-C_{\mathcal{B}}, C_{\mathcal{B}}]$ and (2.3) implies that

$$|f_0(x,u) - f_0(x,v)| \le L_{\mathcal{B}}|u-v|,\ x \in \mathbb{R}^N, \tag{2.4}$$

with $L_{\mathcal{B}} > 0$ being a multiple of $1 + 2C_{\mathcal{B}}^{\rho-1}$.

The proof of Theorem 1.2 now follows in a sequence of lemmas.

Lemma 2.3. *Assume conditions of Theorem 1.1 and let $\mathcal{B}$ be as in (1.17). Then for arbitrarily fixed $T > 0$ and $\alpha \in (-\frac{3}{4}, \frac{1}{4}]$ there is a positive constant $\mu = \mu_{\alpha,T}$ such that for any $u_0, v_0 \in \mathcal{B}$*

$$\|S(t)u_0 - S(t)v_0\|_{H^1(\mathbb{R}^N)} \le \frac{\mu}{t^{\frac{1}{4}-\alpha}}\|u_0 - v_0\|_{H^{4\alpha}(\mathbb{R}^N)}, \quad t \in (0,T].$$

Proof. As follows from considerations of [14, Appendix B] the solution $u = S(\cdot)u_0$ of (1.1) through $u_0 \in H^1(\mathbb{R}^N)$ satisfies Duhamel's formula

$$u(t) = e^{-\Delta^2 t}u_0 + \int_0^t e^{-\Delta^2(t-s)}(-\Delta(f(\cdot,u(s)) - \delta u(s))ds, \quad t > 0. \tag{2.5}$$

Writing (2.5) for $u = S(\cdot)u_0$ and $v = S(\cdot)v_0$ respectively, letting

$$\mathfrak{F}(u,v) := f(\cdot,u) - f(\cdot,v) = m(\cdot)(u-v) + f_0(\cdot,u) - f_0(\cdot,v)$$

and using the estimates of $\{e^{-\Delta^2 t}\}_{t\ge 0}$ in Bessel scale as in [13, Corollary 2.7], we have that $U = u - v$ satisfies for a given $\varepsilon \in (0, \frac{1}{4})$ and $0 < t \le T$

$$\|U(t)\|_{H^1(\mathbb{R}^N)} \le \frac{M_T}{t^{\frac{1}{4}-\alpha}}\|U(0)\|_{H^{4\alpha}(\mathbb{R}^N)} + \int_0^t \frac{M_T}{(t-s)^{\frac{3}{4}+\varepsilon}}\mathcal{R}(s)ds,$$

where M_T is a certain positive constant and

$$\mathcal{R}(s) = \| -\delta U(s) - \Delta(\mathfrak{F}(u(s),v(s)))\|_{(H^{2+4\varepsilon}(\mathbb{R}^N))'}.$$

Following [53], we remark that for any real θ, $\tilde{\theta}$ we have

$$(H^\theta(\mathbb{R}^N))' = H^{-\theta}(\mathbb{R}^N) \quad \text{and} \quad H^\theta(\mathbb{R}^N) \subset H^{\tilde{\theta}}(\mathbb{R}^N) \ \text{ when } \theta \ge \tilde{\theta}. \tag{2.6}$$

We frequently use that

$$\|\phi\|_{L^2(\mathbb{R}^N)} \le \|\phi\|_{H^1(\mathbb{R}^N)},\ \phi \in H^1(\mathbb{R}^N). \tag{2.7}$$

Also, due to [53, formula (1) on page 177 and Step 2 atop page 191]

$$\|\cdot\|_{H^{-2}(\mathbb{R}^N)} = \|(-\Delta+1)^{-1}\cdot\|_{L^2(\mathbb{R}^N)}, \tag{2.8}$$

whereas from [47, formula (7.1) in Chapter 1] and contractive property of the heat semigroup

$$\|(-\Delta+1)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^N))} = \|\int_0^\infty e^{-t}e^{\Delta t}dt\|_{L^2(\mathbb{R}^N)} \leq 1, \tag{2.9}$$

From (2.6) it is straightforward that $\|U\|_{(H^{2+4\varepsilon}(\mathbb{R}^N))'}$ is bounded by a multiple of $\|U\|_{H^1(\mathbb{R}^N)}$. Choosing $\varepsilon \in (0,\frac{1}{4})$ close enough to $\frac{1}{4}$, we combine (2.6) with [16, Lemma 5.2] to get

$$\|\Delta(m(\cdot)U)\|_{(H^{2+4\varepsilon}(\mathbb{R}^N))'} \leq C_{m,\varepsilon}\|U\|_{H^1(\mathbb{R}^N)}$$

with some constant $C_{m,\varepsilon}$. Using again (2.6) we observe that $\|\Delta(f_0(\cdot,u)-f_0(\cdot,v))\|_{(H^{2+4\varepsilon}(\mathbb{R}^N))'}$ is bounded by a multiple of $\|\Delta(f_0(\cdot,u)-f_0(\cdot,v))\|_{H^{-2}(\mathbb{R}^N)}$, which by (2.8) and (2.9) is, in turn, bounded by a multiple of $\|f_0(\cdot,u)-f_0(\cdot,v)\|_{L^2(\mathbb{R}^N)}$. Since due to (2.4)

$$\|f_0(\cdot,u)-f_0(\cdot,v)\|_{L^2(\mathbb{R}^N)} \leq L_{\mathcal{B}}\|U\|_{L^2(\mathbb{R}^N)} \tag{2.10}$$

we actually have

$$\mathcal{R}(s) = \|-\delta U(s) - \Delta(\mathfrak{F}(u(s),v(s)))\|_{(H^{2+4\varepsilon}(\mathbb{R}^N))'} \leq C_{m,\mathcal{B},\varepsilon}\|U(s)\|_{H^1(\mathbb{R}^N)}$$

for some constant $C_{m,\mathcal{B},\varepsilon} > 0$ and we thus obtain

$$\|U(t)\|_{H^1(\mathbb{R}^N)} \leq \frac{M_T}{t^{\frac{1}{4}-\alpha}}\|U(0)\|_{H^{4\alpha}(\mathbb{R}^N)} + \int_0^t \frac{C_{m,\mathcal{B},\varepsilon}M_T}{(t-s)^{\frac{3}{4}+\varepsilon}}\|U(s)\|_{H^1(\mathbb{R}^N)}ds,\ 0<t\leq T.$$

From this and [12, Lemma 1.2.9] we get the result. □

In the next lemma we derive an $L^2(0,T;H^2(\mathbb{R}^N))$-estimate of the difference of solutions.

Lemma 2.4. *Under the conditions and notation of Lemma 2.3 if $u_0, v_0 \in \mathcal{B}$ then given any $T>0$ there is a constant $\kappa_T > 0$ such that $U = S(\cdot)u_0 - S(\cdot)v_0$ satisfies*

$$\|U\|_{L^2(0,T;H^2(\mathbb{R}^N))} \leq \kappa_T\|U(0)\|_{H^1(\mathbb{R}^N)}.$$

Proof. Writing (1.1) for $u = S(t)u_0$, $v = S(t)v_0$, we observe that $U = u - v$ satisfies

$$U_t + \Delta(\Delta U + \mathfrak{F}(u,v)) + \delta U = 0.$$

From this for any $\gamma > 0$ we get

$$(-\Delta+\gamma)^{-1}U_t - \Delta U - \mathfrak{F}(u,v) - \gamma U + \gamma(-\Delta+\gamma)^{-1}\mathfrak{F}(u,v) + (\gamma^2+\delta)(-\Delta+\gamma)^{-1}U = 0. \tag{2.11}$$

Writing (2.11) with $\gamma = 1$, multiplying by ΔU in $L^2(\mathbb{R}^N)$ and taking into account that

$$\begin{aligned}\langle(-\Delta+1)^{-1}U_t,\Delta U\rangle_{L^2(\mathbb{R}^N)} &= \langle U_t,(-\Delta+1)^{-1}\Delta U\rangle_{L^2(\mathbb{R}^N)}\\ &= \langle U_t,-U+(-\Delta+1)^{-1}U\rangle_{L^2(\mathbb{R}^N)}\\ &= \frac{1}{2}\frac{d}{dt}(-\|U\|^2_{L^2(\mathbb{R}^N)} + \|(-\Delta+1)^{-\frac{1}{2}}U\|^2_{L^2(\mathbb{R}^N)}),\end{aligned}$$

we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(-\|U\|^2_{L^2(\mathbb{R}^N)} + \|(-\Delta+1)^{-\frac{1}{2}}U\|^2_{L^2(\mathbb{R}^N)}) - \|\Delta U\|^2_{L^2(\mathbb{R}^N)}\\ &\quad - \langle\mathfrak{F}(u,v),\Delta U\rangle_{L^2(\mathbb{R}^N)} - \langle U,\Delta U\rangle_{L^2(\mathbb{R}^N)} + \langle(-\Delta+1)^{-1}\mathfrak{F}(u,v),\Delta U\rangle_{L^2(\mathbb{R}^N)}\\ &\qquad + (1+\delta)\langle(-\Delta+1)^{-1}U,\Delta U\rangle_{L^2(\mathbb{R}^N)} = 0.\end{aligned}$$

From (2.9) and (2.10) we have

$$\|(-\Delta+1)^{-1}(f_0(\cdot,u)-f_0(\cdot,v))\|_{L^2(\mathbb{R}^N)} \le L_{\mathcal{B}}\|U\|_{L^2(\mathbb{R}^N)},$$

whereas from (2.8) and [16, Lemma 5.2], we get

$$\|(-\Delta+1)^{-1}(m(\cdot)U)\|_{L^2(\mathbb{R}^N)} = \|m(\cdot)U\|_{H^{-2}(\mathbb{R}^N)} \le C_m(\|U\|^2_{L^2(\mathbb{R}^N)} + \|\nabla U\|^2_{L^2(\mathbb{R}^N)})^{\frac{1}{2}}$$

with some constant $C_m > 0$, so that

$$\|(-\Delta+1)^{-1}\mathfrak{F}(u,v)\|^2_{L^2(\mathbb{R}^N)} \le 2C_m^2(\|U\|^2_{L^2(\mathbb{R}^N)} + \|\nabla U\|^2_{L^2(\mathbb{R}^N)}) + 2L^2_{\mathcal{B}}\|U\|^2_{L^2(\mathbb{R}^N)}. \tag{2.12}$$

Then the Cauchy-Schwarz inequality used with (2.12) implies for any $\varepsilon > 0$

$$|\langle(-\Delta+1)^{-1}\mathfrak{F}(u,v),\Delta U\rangle_{L^2(\mathbb{R}^N)}| \le \frac{1}{\varepsilon}(C_m^2 + L^2_{\mathcal{B}})\|U\|^2_{H^1(\mathbb{R}^N)} + \frac{\varepsilon}{2}\|\Delta U\|^2_{L^2(\mathbb{R}^N)},$$

whereas with (2.10) yields

$$|\langle f_0(x,u)-f_0(x,v),\Delta U\rangle_{L^2(\mathbb{R}^N)}| \le \frac{1}{2\varepsilon}L^2_{\mathcal{B}}\|U\|^2_{L^2(\mathbb{R}^N)} + \frac{\varepsilon}{2}\|\Delta U\|^2_{L^2(\mathbb{R}^N)}.$$

Since for $\alpha \in (0,1)$ close enough to 1 due to [14, Lemma A.1] we have

$$\|m(\cdot)U\|_{L^2(\mathbb{R}^N)} \le C_{m,\alpha}\|U\|_{H^{2\alpha}(\mathbb{R}^N)}, \tag{2.13}$$

using interpolation inequality $\|U\|_{H^{2\alpha}(\mathbb{R}^N)} \le c_\alpha \|U\|^{\alpha}_{H^2(\mathbb{R}^N)}\|U\|^{1-\alpha}_{L^2(\mathbb{R}^N)}$ (see [53, (11) on page 185]), we obtain by the Young inequality that

$$\|m(\cdot)U\|_{L^2(\mathbb{R}^N)} \le \varepsilon\|\Delta U\|_{L^2(\mathbb{R}^N)} + d_\varepsilon\|U\|_{L^2(\mathbb{R}^N)}$$

where we have also used that $\|\cdot\|_{L^2(\mathbb{R}^N)} + \|\Delta\cdot\|_{L^2(\mathbb{R}^N)}$ is an equivalent norm in $H^2(\mathbb{R}^N)$. With the Cauchy-Schwarz inequality this yields

$$\begin{aligned}
|\langle m(\cdot)U,\Delta U\rangle_{L^2(\mathbb{R}^N)}| &\le \|m(\cdot)U\|_{L^2(\mathbb{R}^N)}\|\Delta U\|_{L^2(\mathbb{R}^N)} \\
&\le \varepsilon\|\Delta U\|^2_{L^2(\mathbb{R}^N)} + d_\varepsilon\|U\|_{L^2(\mathbb{R}^N)}\|\Delta U\|_{L^2(\mathbb{R}^N)} \\
&\le \frac{3\varepsilon}{2}\|\Delta U\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2\varepsilon}d_\varepsilon^2\|U\|^2_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

Summarizing, we get

$$|\langle\mathfrak{F}(u,v),\Delta U\rangle_{L^2(\mathbb{R}^N)}| \le 2\varepsilon\|\Delta U\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2\varepsilon}(d_\varepsilon^2 + L^2_{\mathcal{B}})\|U\|^2_{L^2(\mathbb{R}^N)}.$$

Note that due to the Cauchy-Schwarz inequality and (2.9) we also have

$$|\langle(-\Delta+1)^{-1}U,\Delta U\rangle_{L^2(\mathbb{R}^N)}| \le \frac{1}{2\varepsilon}\|U\|^2_{L^2(\mathbb{R}^N)} + \frac{\varepsilon}{2}\|\Delta U\|^2_{L^2(\mathbb{R}^N)}.$$

Similarly,

$$|\langle U,\Delta U\rangle_{L^2(\mathbb{R}^N)}| \le \frac{1}{2\varepsilon}\|U\|^2_{L^2(\mathbb{R}^N)} + \frac{\varepsilon}{2}\|\Delta U\|^2_{L^2(\mathbb{R}^N)}.$$

As a consequence, taking also into account (2.7), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(-\|U\|^2_{L^2(\mathbb{R}^N)} + \|(-\Delta+1)^{-\frac{1}{2}}U\|^2_{L^2(\mathbb{R}^N)}) - \|\Delta U\|^2_{L^2(\mathbb{R}^N)} \\
&\qquad + \frac{\varepsilon}{2}(7+\delta)\|\Delta U\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{\varepsilon}\left(1+\frac{\delta}{2}+C_m^2+\frac{d_\varepsilon^2}{2}+\frac{3L^2_{\mathcal{B}}}{2}\right)\|U\|^2_{H^1(\mathbb{R}^N)} \ge 0.
\end{aligned}$$

Choosing now $\varepsilon = \frac{1}{7+\delta}$ and denoting $d := 2(7+\delta)(1+\frac{\delta}{2}+C_m^2+\frac{d_\varepsilon^2}{2}+\frac{3L_{\mathcal{B}}^2}{2})$, we get

$$\frac{d}{dt}(-\|U\|^2_{L^2(\mathbb{R}^N)} + \|(-\Delta+1)^{-\frac{1}{2}}U\|^2_{L^2(\mathbb{R}^N)}) + d\|U\|^2_{H^1(\mathbb{R}^N)} \geq \|\Delta U\|^2_{L^2(\mathbb{R}^N)}.$$

Integrating with respect to $t \in [0,T]$, then omitting on the left-hand side the negative terms resulting from integration and using (2.1), we obtain

$$\|U(0)\|^2_{L^2(\mathbb{R}^N)} + \|U(T)\|^2_{L^2(\mathbb{R}^N)} + d\|U\|^2_{L^2(0,T;H^1(\mathbb{R}^N))} \geq \|\Delta U\|^2_{L^2(0,T;L^2(\mathbb{R}^N))}.$$

Taking into account (2.7) and applying Lemma 2.3 with $\alpha = \frac{1}{4}$, we have

$$\|U(t)\|^2_{L^2(\mathbb{R}^N)} \leq \|U(t)\|^2_{H^1(\mathbb{R}^N)} \leq \mu^2 \|U(0)\|^2_{H^1(\mathbb{R}^N)},\ t \in (0,T],$$

which after integrating with respect to $t \in (0,T)$ yields

$$\|U\|^2_{L^2(0,T;L^2(\mathbb{R}^N))} \leq \|U\|^2_{L^2(0,T;H^1(\mathbb{R}^N))} \leq \mu^2 T \|U(0)\|^2_{H^1(\mathbb{R}^N)}.$$

As a consequence we get

$$\|\Delta U\|^2_{L^2(0,T;L^2(\mathbb{R}^N))} \leq (1+\mu^2+d\mu^2 T)\|U(0)\|^2_{H^1(\mathbb{R}^N)}.$$

Since $\|\cdot\|_{L^2(\mathbb{R}^N)} + \|\Delta\cdot\|_{L^2(\mathbb{R}^N)}$ is an equivalent norm in $H^2(\mathbb{R}^N)$, the result now follows easily. $\square$

Using Lemma 2.4 we derive below an estimate involving time derivative.

Lemma 2.5. *Under the conditions and notation of Lemma 2.3 if $u_0, v_0 \in \mathcal{B}$ then given any $T>0$ there is a constant $\omega_T > 0$ such that $U = S(\cdot)u_0 - S(\cdot)v_0$ satisfies*

$$\|U_t\|_{L^2(0,T;H^{-2}(\mathbb{R}^N))} \leq \omega_T \|U(0)\|_{H^1(\mathbb{R}^N)}.$$

Proof. Writing (2.11) with $\gamma = 1$ and applying $L^2(\mathbb{R}^N)$-norm, we get

$$\begin{aligned}\|(-\Delta+1)^{-1}U_t\|_{L^2(\mathbb{R}^N)} \leq{}& \|\Delta U\|_{L^2(\mathbb{R}^N)} + \|\mathfrak{F}(u,v)\|_{L^2(\mathbb{R}^N)} + \|U\|_{L^2(\mathbb{R}^N)} \\ &+ \|(-\Delta+1)^{-1}\mathfrak{F}(u,v)\|_{L^2(\mathbb{R}^N)} + (1+\delta)\|(-\Delta+1)^{-1}U\|_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Due to (2.9), (2.10) and (2.13) both $\|(-\Delta+1)^{-1}\mathfrak{F}(u,v)\|_{L^2(\mathbb{R}^N)}$ and $\|\mathfrak{F}(u,v)\|_{L^2(\mathbb{R}^N)}$ are bounded from above by a multiple of $\|U\|_{H^2(\mathbb{R}^N)}$ (see the embedding in (2.6)). It is also straightforward that the sum $\|\Delta U\|_{L^2(\mathbb{R}^N)} + \|U\|_{L^2(\mathbb{R}^N)} + (1+\delta)\|(-\Delta+1)^{-1}U\|_{L^2(\mathbb{R}^N)}$ is estimated from above by a multiple of $\|U\|_{H^2(\mathbb{R}^N)}$ (see again (2.9)). As a consequence there is a constant $\omega > 0$ such that

$$\|(-\Delta+1)^{-1}U_t\|_{L^2(\mathbb{R}^N)} \leq \omega \|U\|_{H^2(\mathbb{R}^N)},\ t>0,$$

which in turn yields

$$\|(-\Delta+1)^{-1}U_t\|_{L^2(0,T;L^2(\mathbb{R}^N))} \leq \omega \|U\|_{L^2(0,T;H^2(\mathbb{R}^N))},\ T>0.$$

Using now (2.8) and Lemma 2.4, we get the result. $\square$

Given a positive function φ of variable $x \in \mathbb{R}^N$ and $q \geq 1$, we denote by $L^q_\varphi(\mathbb{R}^N)$ the weighted space consisting of all $\phi \in L^q_{loc}(\mathbb{R}^N)$ such that $\phi\varphi^{\frac{1}{q}} \in L^q(\mathbb{R}^N)$, with norm

$$\|\phi\|_{L^q_\varphi(\mathbb{R}^N)} = \|\phi\varphi^{\frac{1}{q}}\|_{L^q(\mathbb{R}^N)}.$$

In what follows, we consider $\varphi : \mathbb{R}^N \to (0,1]$ of the form

$$\varphi(\cdot) = \frac{1}{(1+|\cdot|^2)^{\varphi_0}} \quad \text{with} \quad \varphi_0 > \frac{N}{2}. \tag{2.14}$$

Lemma 2.6. *Under the conditions and notation of Lemma 2.3 there exist positive constants τ and γ such that if $u_0, v_0 \in \mathcal{B}$ then $U = S(\cdot)u_0 - S(\cdot)v_0$ satisfies for each $t \geq \tau$ and all sufficiently large R the estimate of the form*

$$\|(-\Delta+\gamma)^{-\frac{1}{2}}U(t)\|_{L^2(\mathbb{R}^N)} \leq e^{-\frac{\delta}{2}(t-\tau)}\|(-\Delta+\gamma)^{-\frac{1}{2}}U(\tau)\|_{L^2(\mathbb{R}^N)} + \zeta_R\|U\|_{L^2(\tau,t;L^{2r'}_\varphi(\mathbb{R}^N))} + \alpha_R\|U\|_{L^2(\tau,t;L^{2s'}(\mathbb{R}^N))}$$

where φ is as in (2.14), r is from (1.5), ζ_R is a certain constant depending on R and $\alpha_R \to 0^+$ as $R \to \infty$.

Proof. For $u = S(t)u_0$, $v = S(t)v_0$ with $u_0, v_0 \in \mathcal{B}$, multiplying (2.11) by $U = u - v$ in $L^2(\mathbb{R}^N)$ gives for any $\gamma > 0$

$$\frac{1}{2}\frac{d}{dt}\|(-\Delta+\gamma)^{-\frac{1}{2}}U\|^2_{L^2(\mathbb{R}^N)} + \|\nabla U\|^2_{L^2(\mathbb{R}^N)} - \langle \mathfrak{F}(u,v), U\rangle_{L^2(\mathbb{R}^N)} - \gamma\|U\|^2_{L^2(\mathbb{R}^N)} + (\gamma^2+\delta)\|(-\Delta+\gamma)^{-\frac{1}{2}}U\|^2_{L^2(\mathbb{R}^N)} = -\gamma\langle(-\Delta+\gamma)^{-1}\mathfrak{F}(u,v), U\rangle_{L^2(\mathbb{R}^N)}$$

and we proceed in three steps.

Step 1. In this step we find an estimate of $\|\nabla U\|^2_{L^2(\mathbb{R}^N)} - \langle \mathfrak{F}(u,v), U\rangle_{L^2(\mathbb{R}^N)}$ from below.

By assumption there exists $\omega_0 > 0$ such that (1.9) holds and there is a continuous decreasing real valued function $\omega(\nu)$ defined in a certain interval $[0, \nu_0]$ and satisfying

$$\lim_{\nu\to 0^+} \omega(\nu) = \omega(0) = \omega_0$$

such that

$$\int_{\mathbb{R}^N} \big((1-\nu)|\nabla\phi|^2 - C(x)\phi^2\big) \geq \omega(\nu)\|\phi\|^2_{L^2(\mathbb{R}^N)} \quad \text{for} \quad \nu \in [0,\nu_0],\ \phi \in H^1(\mathbb{R}^N).$$

(see [14, Theorems A.2 and A.4]). Using this with a suitably small $\nu > 0$ (such ν is fixed from now on), and applying (2.2), given any ball $B_R = \{x \in \mathbb{R}^N : |x| < R\}$ we get

$$\begin{aligned}\int_{\mathbb{R}^N}(|\nabla U|^2 - m(x)U^2) &= \int_{\mathbb{R}^N}(\nu+1-\nu)|\nabla U|^2 - \int_{\mathbb{R}^N}(C+m-C)(x)U^2 \\ &\geq \nu\|\nabla U\|^2_{L^2(\mathbb{R}^N)} + \frac{3\omega_0}{4}\|U\|^2_{L^2(\mathbb{R}^N)} - \Big(\int_{B_R} + \int_{B_R^c}\Big)(m-C)(x)U^2 \\ &\geq \nu\|\nabla U\|^2_{L^2(\mathbb{R}^N)} + \frac{\omega_0}{2}\|U\|^2_{L^2(\mathbb{R}^N)} - \int_{B_R}(m-C)(x)U^2 - \alpha_0\int_{B_R^c} D(x)U^2\end{aligned}$$

where $B_R^c = \mathbb{R}^N \setminus B_R$. Also, by assumption on D, due to the Hölder inequality we have

$$\int_{B_R^c} D(x)U^2 \leq \|D\|_{L^s(B_R^c)}\|U\|^2_{L^{2s'}(\mathbb{R}^N)}.$$

Since by the locally uniform integrability of m and C there is a positive constant k_R such that

$$\int_{B_R}(m(x) - C(x))U^2 \leq k_R\|U\|^2_{L^{2r'}(B_R)},$$

where the exponent r' is Hölder's conjugate to r, we actually get

$$\int_{\mathbb{R}^N}(|\nabla U|^2 - m(x)U^2) \geq \nu\|\nabla U\|^2_{L^2(\mathbb{R}^N)} + \frac{\omega_0}{2}\|U\|^2_{L^2(\mathbb{R}^N)} - k_R\|U\|^2_{L^{2r'}(B_R)} - \alpha_0\|D\|_{L^s(B_R^c)}\|U\|^2_{L^{2s'}(\mathbb{R}^N)}.$$

Applying (2.3) and (2.4), we get

$$\int_{\mathbb{R}^N}(f_0(x,u)-f_0(x,v))U \le \Big(\int_{B_R}+\int_{B_R^c}\Big)|f_0(x,u)-f_0(x,v)||U|$$
$$\le L_{\mathcal{B}}\int_{B_R}U^2+\frac{\omega_0}{8}\int_{B_R^c}U^2+c_*\int_{B_R^c}(|u|^{\rho-1}+|v|^{\rho-1})U^2.$$

Due to (1.18) given any $\xi>0$ there exist $\tau_\xi>0$ and $R_\xi>0$ such that $\sup_{t\ge\tau_\xi}\|u\|_{L^2(B^c_{R_\xi})}\le\xi$ which yields, since values of u are within the range of $[-C_{\mathcal{B}},C_{\mathcal{B}}]$,

$$\|u\|_{L^\ell(B_R^c)}\le C_{\mathcal{B}}^{1-\frac{2}{\ell}}\|u\|^{\frac{2}{\ell}}_{L^2(B_R^c)}\le C_{\mathcal{B}}^{1-\frac{2}{\ell}}\xi^{\frac{2}{\ell}} \quad \text{for any} \quad t\ge\tau_\xi \quad \text{and } R\ge R_\xi,\ \ell\ge 2$$

and the same holds true for v. Using this with $\ell=q(\rho-1)$ and $q\ge\max\{\frac{N}{2},\frac{2}{\rho-1}\}$, via the Hölder inequality and Sobolev embedding $H^1(\mathbb{R}^N)\subset L^{2q'}(\mathbb{R}^N)$, we obtain

$$\int_{B_R^c}(|u|^{\rho-1}+|v|^{\rho-1})U^2\le \||u|^{\rho-1}+|v|^{\rho-1}\|_{L^q(B_R^c)}\|U\|^2_{L^{2q'}(B_R^c)}$$
$$\le c_{\mathrm{S}}^2(q')(\|u\|^{\rho-1}_{L^{q(\rho-1)}(B_R^c)}+\|v\|^{\rho-1}_{L^{q(\rho-1)}(B_R^c)})\|U\|^2_{H^1(\mathbb{R}^N)}$$
$$\le 2c_{\mathrm{S}}^2(q')C_{\mathcal{B}}^{\rho-1-\frac{2}{q}}\xi^{\frac{2}{q}}(\|U\|^2_{L^2(\mathbb{R}^N)}+\|\nabla U\|^2_{L^2(\mathbb{R}^N)}),\ t\ge\tau_\xi,\ R\ge R_\xi,$$

where $c_{\mathrm{S}}(q')$ denotes the Sobolev embedding constant. Choosing then suitably small $\xi>0$, we get for all large enough R

$$\int_{\mathbb{R}^N}(f_0(x,u)-f_0(x,v))U\le L_{\mathcal{B}}\int_{B_R}U^2+\frac{\omega_0}{8}\|U\|^2_{L^2(\mathbb{R}^N)}$$
$$+\min\left\{\frac{3\nu}{4},\frac{\omega_0}{8}\right\}(\|U\|^2_{L^2(\mathbb{R}^N)}+\|\nabla U\|^2_{L^2(\mathbb{R}^N)}),\ t\ge\tau_\xi.$$

As a consequence we obtain

$$\|\nabla U\|^2_{L^2(\mathbb{R}^N)}-\langle\mathfrak{F}(u,v),U\rangle_{L^2(\mathbb{R}^N)}\ge\frac{\nu}{4}\|\nabla U\|^2_{L^2(\mathbb{R}^N)}+\frac{\omega_0}{4}\|U\|^2_{L^2(\mathbb{R}^N)}-k_R\|U\|^2_{L^{2r'}(B_R)}$$
$$-L_{\mathcal{B}}\|U\|^2_{L^2(B_R)}-\alpha_0\|D\|_{L^s(B_R^c)}\|U\|^2_{L^{2s'}(\mathbb{R}^N)},\ t\ge\tau_\xi.$$

Step 2. In this step we find an estimate of $|\gamma\langle(-\Delta+\gamma)^{-1}\mathfrak{F}(u,v),U\rangle_{L^2(\mathbb{R}^N)}|$ from above.

Since due to the resolvent equation

$$(-\Delta+\gamma)^{-1}\mathfrak{F}(u,v)=(-\Delta+1)^{-1}\mathfrak{F}(u,v)+(1-\gamma)(-\Delta+\gamma)^{-1}(-\Delta+1)^{-1}\mathfrak{F}(u,v),$$

we have

$$|\gamma\langle(-\Delta+\gamma)^{-1}\mathfrak{F}(u,v),U\rangle_{L^2(\mathbb{R}^N)}|\le|\gamma\langle(-\Delta+1)^{-1}\mathfrak{F}(u,v),U\rangle_{L^2(\mathbb{R}^N)}|$$
$$+|\gamma\langle(1-\gamma)(-\Delta+\gamma)^{-\frac{1}{2}}(-\Delta+1)^{-1}\mathfrak{F}(u,v),(-\Delta+\gamma)^{-\frac{1}{2}}U\rangle_{L^2(\mathbb{R}^N)}|.$$

From this, using the Cauchy-Schwarz inequality with $\varepsilon=1$ and $\varepsilon=\frac{\gamma^2+\delta}{\gamma}$ respectively, we get

$$|\gamma\langle(-\Delta+\gamma)^{-1}\mathfrak{F}(u,v),U\rangle_{L^2(\mathbb{R}^N)}|\le\frac{\gamma}{2}\|(-\Delta+1)^{-1}\mathfrak{F}(u,v)\|^2_{L^2(\mathbb{R}^N)}+\frac{\gamma}{2}\|U\|^2_{L^2(\mathbb{R}^N)}$$
$$+\frac{\gamma^2+\delta}{2}\|(-\Delta+\gamma)^{-\frac{1}{2}}U\|^2_{L^2(\mathbb{R}^N)}+\frac{\gamma(1-\gamma)^2}{2(\gamma^2+\delta)}\|(-\Delta+1)^{-1}\mathfrak{F}(u,v)\|^2_{L^2(\mathbb{R}^N)},$$

where in the last term above we have also used (2.1). Recalling then (2.12), we obtain

$$\begin{aligned}|\gamma\langle(-\Delta+\gamma)^{-1}\mathfrak{F}(u,v),U\rangle_{L^2(\mathbb{R}^N)}| &\le \gamma\Big(1+\frac{(1-\gamma)^2}{\gamma^2+\delta}\Big)C_m^2(\|U\|^2_{L^2(\mathbb{R}^N)}+\|\nabla U\|^2_{L^2(\mathbb{R}^N)})\\ &+\gamma\left(\frac12+L_{\mathcal{B}}^2+\frac{(1-\gamma)^2}{\gamma^2+\delta}L_{\mathcal{B}}^2\right)\|U\|^2_{L^2(\mathbb{R}^N)}+\frac{\gamma^2+\delta}{2}\|(-\Delta+\gamma)^{-\frac12}U\|^2_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Step 3. We now combine the estimates to complete the proof.

Hence for all $t\ge\tau_\xi$, any $\gamma>0$ and each large enough R we have

$$\begin{aligned}&\frac12\frac{d}{dt}\|(-\Delta+\gamma)^{-\frac12}U\|^2_{L^2(\mathbb{R}^N)}+\frac{\nu}{4}\|\nabla U\|^2_{L^2(\mathbb{R}^N)}+\frac{\omega_0}{4}\|U\|^2_{L^2(\mathbb{R}^N)}-k_R\|U\|^2_{L^{2r'}(B_R)}-L_{\mathcal{B}}\|U\|^2_{L^2(B_R)}\\ &\quad-\gamma\|U\|^2_{L^2(\mathbb{R}^N)}-\alpha_0\|D\|_{L^s(B_R^c)}\|U\|^2_{L^{2s'}(\mathbb{R}^N)}+\frac{\gamma^2+\delta}{2}\|(-\Delta+\gamma)^{-\frac12}U\|^2_{L^2(\mathbb{R}^N)}\\ &\le\gamma\Big(1+\frac{(1-\gamma)^2}{\gamma^2+\delta}\Big)C_m^2(\|U\|^2_{L^2(\mathbb{R}^N)}+\|\nabla U\|^2_{L^2(\mathbb{R}^N)})\\ &\quad+\gamma\left(\frac12+L_{\mathcal{B}}^2+\frac{(1-\gamma)^2}{\gamma^2+\delta}L_{\mathcal{B}}^2\right)\|U\|^2_{L^2(\mathbb{R}^N)},\end{aligned}$$

which with $\gamma>0$ chosen now suitably small gives

$$\begin{aligned}&\frac12\frac{d}{dt}\|(-\Delta+\gamma)^{-\frac12}U\|^2_{L^2(\mathbb{R}^N)}+\frac{\nu}{8}\|\nabla U\|^2_{L^2(\mathbb{R}^N)}+\frac{\gamma^2+\delta}{2}\|(-\Delta+\gamma)^{-\frac12}U\|^2_{L^2(\mathbb{R}^N)}\\ &+\frac{\omega_0}{16}\|U\|^2_{L^2(\mathbb{R}^N)}\le k_R\|U\|^2_{L^{2r'}(B_R)}+L_{\mathcal{B}}\|U\|^2_{L^2(B_R)}+\alpha_0\|D\|_{L^s(B_R^c)}\|U\|^2_{L^{2s'}(\mathbb{R}^N)}\\ &\le(k_R+L_{\mathcal{B}}|B_R|^{\frac1r})\|U\|^2_{L^{2r'}(B_R)}+\alpha_0\|D\|_{L^s(B_R^c)}\|U\|^2_{L^{2s'}(\mathbb{R}^N)}.\end{aligned}\tag{2.15}$$

Since $\int_{B_R}|U|^{2r'}\le\int_{B_R}|U|^{2r'}\frac{\varphi}{\inf_{B_R}\varphi}\le\frac{1}{\inf_{B_R}\varphi}\int_{\mathbb{R}^N}|U|^{2r'}\varphi$, we have

$$\|U\|^2_{L^{2r'}(B_R)}\le(\inf_{B_R}\varphi)^{-\frac{1}{r'}}\|U\|^2_{L^{2r'}_\varphi(\mathbb{R}^N)}.$$

Using this in (2.15), denoting $\zeta_R^2=2(k_R+L_{\mathcal{B}}|B_R|^{\frac1r})(\inf_{B_R}\varphi)^{-\frac{1}{r'}}$, and omitting on the left-hand side of (2.15) nonnegative terms $\frac{\nu}{8}\|\nabla U\|^2_{L^2(\mathbb{R}^N)}$, $\frac{\omega_0}{16}\|U\|^2_{L^2(\mathbb{R}^N)}$ and $\frac12\gamma^2\|(-\Delta+\gamma)^{-\frac12}U\|^2_{L^2(\mathbb{R}^N)}$, we obtain after letting

$$\alpha_R^2:=2\alpha_0\|D\|_{L^s(B_R^c)}$$

that

$$\frac{d}{dt}\|(-\Delta+\gamma)^{-\frac12}U\|^2_{L^2(\mathbb{R}^N)}+\delta\|(-\Delta+\gamma)^{-\frac12}U\|^2_{L^2(\mathbb{R}^N)}\le\zeta_R^2\|U\|^2_{L^{2r'}_\varphi(\mathbb{R}^N)}+\alpha_R^2\|U\|^2_{L^{2s'}(\mathbb{R}^N)},\ t\ge\tau_\xi.$$

The result now follows easily. $\square$

Then we have the following result.

Lemma 2.7. *Under the conditions and notation of Lemma 2.3 if τ and γ are as in Lemma 2.6 and $u_0,v_0\in S(\tau)\mathcal{B}$ then $U=S(\cdot)u_0-S(\cdot)v_0$ satisfies for each $T\ge0$ and all sufficiently large R the estimate*

$$\begin{aligned}\|(-\Delta+\gamma)^{-\frac12}U(T)\|_{L^2(\mathbb{R}^N)}&\le e^{-\frac{\delta T}{2}}\|(-\Delta+\gamma)^{-\frac12}U(0)\|_{L^2(\mathbb{R}^N)}+\zeta_R\|U\|_{L^2(0,T;L^{2r'}_\varphi(\mathbb{R}^N))}\\ &\quad+\alpha_R\|U\|_{L^2(0,T;L^{2s'}(\mathbb{R}^N))}\end{aligned}$$

with φ, ζ_R and α_R as in Lemma 2.6.

Proof. By assumption $u_0 = S(\tau)\tilde{u}_0$, $v_0 = S(\tau)\tilde{v}_0$ for some $\tilde{u}_0, \tilde{v}_0 \in \mathcal{B}$, whereas by Lemma 2.6 $\tilde{U}(t) = S(t)\tilde{u}_0 - S(t)\tilde{v}_0$ satisfies for $t \geq \tau$ and all sufficiently large R

$$\|(-\Delta+\gamma)^{-\frac{1}{2}}\tilde{U}(t)\|_{L^2(\mathbb{R}^N)} \leq e^{-\frac{\delta}{2}(t-\tau)}\|(-\Delta+\gamma)^{-\frac{1}{2}}\tilde{U}(\tau)\|_{L^2(\mathbb{R}^N)} + \zeta_R\|\tilde{U}\|_{L^2(\tau,t;L^{2r'}_\varphi(\mathbb{R}^N))} + \alpha_R\|\tilde{U}\|_{L^2(\tau,t;L^{2s'}(\mathbb{R}^N))}.$$

Since $\tilde{U}(t) = U(t-\tau)$ whenever $t-\tau \geq 0$ and $\tilde{U}(\tau) = U(0)$, we actually have for $t \geq \tau$

$$\begin{aligned}\|(-\Delta+\gamma)^{-\frac{1}{2}}U(t-\tau)\|_{L^2(\mathbb{R}^N)} &\leq e^{-\frac{\delta}{2}(t-\tau)}\|(-\Delta+\gamma)^{-\frac{1}{2}}U(0)\|_{L^2(\mathbb{R}^N)} \\ &\quad + \zeta_R\|U(\cdot-\tau)\|_{L^2(\tau,t;L^{2r'}_\varphi(\mathbb{R}^N))} + \alpha_R\|U(\cdot-\tau)\|_{L^2(\tau,t;L^{2s'}(\mathbb{R}^N))}\end{aligned}$$

and letting $T = t-\tau$ we get the result. $\square$

Given τ as in Lemma 2.6 we define

$$\mathbf{B} = \mathrm{cl}_{H^1(\mathbb{R}^N)}\, S(\tau)\mathcal{B}. \tag{2.16}$$

Remark 2.8. From the semigroup property and the properties of $\mathcal{B}$ listed below (1.17) (see also Remark 2.1) it follows that $\mathbf{B} \subset \mathcal{B}$, $\mathbf{B}$ is bounded in $H^2(\mathbb{R}^N)$ and that $\mathbf{B}$ is positively invariant and absorbs bounded sets of $H^1(\mathbb{R}^N)$ under $\{S(t)\}_{t\geq 0}$.

Due to Lemmas 2.3 and 2.4 the estimate in Lemma 2.7 now leads to the following estimate in $H^1(\mathbb{R}^N)$.

Lemma 2.9. *Under the conditions and notation of Lemma 2.3 given any $q_* \in (0,1)$ there exist $t_* \geq 1$ and $\xi_* > 0$ such that if $\mathbf{B}$ is as in (2.16) and $u_0, v_0 \in \mathbf{B}$ then $U = S(\cdot)u_0 - S(\cdot)v_0$ satisfies*

$$\|U(t_*)\|_{H^1(\mathbb{R}^N)} \leq q_*\|U(0)\|_{H^1(\mathbb{R}^N)} + \xi_*\|U\|_{L^2(0,t_*;L^{2r'}_\varphi(\mathbb{R}^N))}, \tag{2.17}$$

where φ is as in (2.14).

Proof. First let $u_0, v_0 \in S(\tau)\mathcal{B}$. Due to positive invariance of $\mathcal{B}$, $S(t)u_0, S(t)v_0 \in \mathcal{B}$ for every $t \geq 0$. Given $t \geq 1$, using the semigroup property and then Lemma 2.3 with $\alpha = -\frac{1}{2}$ and $T = 1$, we get via (2.8)

$$\begin{aligned}\|U(t)\|_{H^1(\mathbb{R}^N)} &= \|S(1)S(t-1)u_0 - S(1)S(t-1)v_0\|_{H^1(\mathbb{R}^N)} \\ &\leq \mu\|S(t-1)u_0 - S(t-1)v_0\|_{H^{-2}(\mathbb{R}^N)} \\ &= \mu\|(-\Delta+1)^{-1}(S(t-1)u_0 - S(t-1)v_0)\|_{L^2(\mathbb{R}^N)} \\ &= \mu\|(-\Delta+1)^{-1}U(t-1)\|_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Due to the resolvent equation and (2.9)

$$\begin{aligned}\|(-\Delta+1)^{-1}U(t-1)\|_{L^2(\mathbb{R}^N)} &= \|(1+(\gamma-1)(-\Delta+1)^{-1})(-\Delta+\gamma)^{-1}U(t-1)\|_{L^2(\mathbb{R}^N)} \\ &\leq (1+|\gamma-1|)\|(-\Delta+\gamma)^{-1}U(t-1)\|_{L^2(\mathbb{R}^N)},\end{aligned}$$

whereas by (2.1)

$$\begin{aligned}\|(-\Delta+\gamma)^{-1}U(t-1)\|_{L^2(\mathbb{R}^N)} &= \|(-\Delta+\gamma)^{-\frac{1}{2}}(-\Delta+\gamma)^{-\frac{1}{2}}U(t-1)\|_{L^2(\mathbb{R}^N)} \\ &\leq \gamma^{-\frac{1}{2}}\|(-\Delta+\gamma)^{-\frac{1}{2}}U(t-1)\|_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Using this and applying the estimate of Lemma 2.7 with $T = t-1$, we get for all sufficiently large R

$$\begin{aligned}\|U(t)\|_{H^1(\mathbb{R}^N)} &\le \mu(1+|\gamma-1|)\gamma^{-\frac12}\|(-\Delta+\gamma)^{-\frac12}U(t-1)\|_{L^2(\mathbb{R}^N)}\\ &\le \mu(1+|\gamma-1|)\gamma^{-\frac12}e^{-\frac{\delta}{2}(t-1)}\|(-\Delta+\gamma)^{-\frac12}U(0)\|_{L^2(\mathbb{R}^N)}\\ &+\mu(1+|\gamma-1|)\gamma^{-\frac12}(\zeta_R\|U\|_{L^2(0,t;L^{2r'}_\varphi(\mathbb{R}^N))}+\alpha_R\|U\|_{L^2(0,t;L^{2s'}(\mathbb{R}^N))}).\end{aligned}$$

Applying (2.1), (2.7) and the Sobolev embedding $H^2(\mathbb{R}^N)\subset L^{2s'}(\mathbb{R}^N)$, we get

$$\begin{aligned}\|U(t)\|_{H^1(\mathbb{R}^N)} &\le \mu(1+|\gamma-1|)\gamma^{-1}e^{-\frac{\delta}{2}(t-1)}\|U(0)\|_{H^1(\mathbb{R}^N)}\\ &+\mu(1+|\gamma-1|)\gamma^{-\frac12}(\zeta_R\|U\|_{L^2(0,t;L^{2r'}_\varphi(\mathbb{R}^N))}+\alpha_R c(s')\|U\|_{L^2(0,t;H^2(\mathbb{R}^N))}),\end{aligned}$$

where $c(s')$ is the Sobolev embedding constant. Applying Lemma 2.4, we thus obtain for each $t\ge 1$ and all sufficiently large R

$$\|U(t)\|_{H^1(\mathbb{R}^N)} \le (\eta_t+\beta_R\kappa_t)\|U(0)\|_{H^1(\mathbb{R}^N)}+\xi_R\|U\|_{L^2(0,t;L^{2r'}_\varphi(\mathbb{R}^N))},$$

with $\xi_R=\mu(1+|\gamma-1|)^{-1}\gamma^{-\frac12}\zeta_R$,

$$\eta_t=\mu(1+|\gamma-1|)\gamma^{-1}e^{-\frac{\delta}{2}(t-1)} \text{ and } \beta_R=\mu(1+|\gamma-1|)^{-1}\gamma^{-\frac12}c(s')\alpha_R.$$

Choosing $t_*\ge 1$ such that $\eta_{t_*}\le \frac{q_*}{2}$, we next take R_* so large that $\beta_{R_*}\kappa_{t_*}\le\frac{q_*}{2}$. Hence we obtain

$$\|U(t_*)\|_{H^1(\mathbb{R}^N)}\le q_*\|U(0)\|_{H^1(\mathbb{R}^N)}+\xi_*\|U\|_{L^2(0,t_*;L^{2r'}_\varphi(\mathbb{R}^N))}, \tag{2.18}$$

where $\xi_*=\xi_{R_*}$.

Taking now $u_0,v_0\in\mathbf{B}$ with $\mathbf{B}$ as in (2.16), we write (2.18) for the approximating sequences of initial data from $S(\tau)\mathcal{B}\subset\mathcal{B}$. Since by Lemma 2.4 and the Sobolev embedding

$$\begin{aligned}\|S(\cdot)u_1-S(\cdot)v_1\|_{L^2(0,t_*;L^{2r'}_\varphi(\mathbb{R}^N))} &\le c\|S(\cdot)u_1-S(\cdot)v_1\|_{L^2(0,t_*;H^2(\mathbb{R}^N))}\\ &\le c\kappa_{t_*}\|u_1-v_1\|_{H^1(\mathbb{R}^N)},\ u_1,v_1\in\mathcal{B},\end{aligned}$$

we can pass to the limit on both sides of (2.18) and get the result. □

We now give the following compactness result.

Lemma 2.10. *Given $T>0$ consider the space*

$$Z=\{z\in L^2(0,T;H^2(\mathbb{R}^N))\colon \frac{dz}{dt}\in L^2(0,T;(H^2(\mathbb{R}^N))')\}$$

with norm $\|z\|_{L^2(0,T;H^2(\mathbb{R}^N))}+\|\frac{dz}{dt}\|_{L^2(0,T;(H^2(\mathbb{R}^N))')}$ and let φ be as in (2.14).

If $p>\max\{\frac{N}{4},1\}$ then Z is compactly embedded in $L^2(0,T;L^{2p'}_\varphi(\mathbb{R}^N))$, where p' is Hölder's conjugate to p. In particular, $n(\cdot)=\|\cdot\|_{L^2(0,T;L^{2p'}_\varphi(\mathbb{R}^N))}$ is a compact norm on Z in the sense of Definition A.1.

Proof. By Sobolev embedding $H^2(\mathbb{R}^N)\subset L^{2p'}(\mathbb{R}^N)$. Taking into account that $\varphi\le 1$ we observe that $L^{2p'}(\mathbb{R}^N)\subset L^{2p'}_\varphi(\mathbb{R}^N)$. Since $\varphi\in L^1(\mathbb{R}^N)$, writing φ as $\varphi^{\frac1p+\frac1{p'}}$ and using the Hölder inequality, we conclude that $L^{2p'}_\varphi(\mathbb{R}^N)\subset L^2_\varphi(\mathbb{R}^N)$, where the latter space is a Hilbert space. Therefore we have

$$H^2(\mathbb{R}^N)\subset L^{2p'}_\varphi(\mathbb{R}^N)\subset L^2_\varphi(\mathbb{R}^N)\subset(L^2_\varphi(\mathbb{R}^N))'\subset(H^2(\mathbb{R}^N))'$$

algebraically and topologically. In addition, the identity map is compact from $H^2(\mathbb{R}^N)$ to $L^{2p'}_{\varphi}(\mathbb{R}^N)$, because $H^2(\mathbb{R}^N)$ is continuously embedded into locally uniform space $H^2_U(\mathbb{R}^N)$, whereas $H^2_U(\mathbb{R}^N)$ is compactly embedded in $L^{2p'}_{\varphi}(\mathbb{R}^N)$ (see [2, Lemma 4.1 (iii)]). Hence [37, Theorem I.5.1] applies and we get the result. □

Proof of Theorem 1.2. We specify that conditions in Definition A.2 hold here with $\mathscr{B} = \mathbf{B}$, $X = H^1(\mathbb{R}^N)$, $Z = \{z \in L^2(0,t_*;H^2(\mathbb{R}^N)) \colon \frac{dz}{dt} \in L^2(0,t_*;(H^2(\mathbb{R}^N))')\}$, $n_Z(\cdot)$ being a multiple of $\|\cdot\|_{L^2(0,t_*;L^{2r'}_{\varphi}(\mathbb{R}^N))}$, and mapping K such that $\mathbf{B} \ni u_0 \overset{K}{\to} S(\cdot)u_0 \in Z$, where t_* is taken from Lemma 2.9. Indeed, with this notation the estimate in (A.2) is satisfied by Lemma 2.9 with $q_* < 1$ and $V = S(t_*)$, whereas compactness of $n_Z(\cdot)$ on the space Z follows from Lemma 2.10. Also, $K \colon \mathscr{B} \to Z$ is globally Lipschitz (cp. (A.1)), since by Lemmas 2.4-2.5 and (2.6)

$$\|Ku_0 - Kv_0\|_Z = \|S(\cdot)u_0 - S(\cdot)v_0\|_Z \le \kappa_* \|u_0 - v_0\|_{H^1(\mathbb{R}^N)}, \quad u_0, v_0 \in \mathbf{B} \tag{2.19}$$

where $\kappa_* = \kappa_{t_*} + \omega_{t_*}$. □

3. Finite-dimensional attractors for the semigroup

First we show Hölder continuity in time of the semigroup $\{S(t)\}_{t\ge0}$.

Lemma 3.1. *Assume conditions of Theorem 1.1 and let B be bounded in $H^2(\mathbb{R}^N)$ and positively invariant under $\{S(t)\}_{t\ge0}$.*

Then, given any $\beta < 2$, for arbitrarily fixed $\zeta \in (0,1)$ and $T > \tau > 0$ we have

$$\|S(t_1)u_0 - S(t_2)u_0\|_{H^\beta(\mathbb{R}^N)} \le \chi |t_1 - t_2|^\zeta, \quad t_1, t_2 \in [\tau, T],\ u_0 \in B \tag{3.1}$$

for some positive constant χ.

Proof. Letting $\mathcal{F}_j(u)$, $j = 1,2,3$, as in [14, Appendix B] and using [14, Lemmas B.1, B.2, B.3] with $p = 2$ and $r \ge 2$, observe that

$$H^{4(\alpha-\frac{1}{2})}(\mathbb{R}^N) \ni u \to \sum_{j=1}^{3} \mathcal{F}_j(u) + const.u \in (H^2(\mathbb{R}^N))'$$

is Lipschitz continuous on bounded sets whenever $\alpha < 1$ is close enough to 1. Concerning $u = S(\cdot)u_0$, observe that it is the solution as in [12, Lemma 2.2.1] if we let therein $F(u) = \sum_{j=1}^{3} \mathcal{F}_j(u) + (\delta+1)u$, $A = \Delta^2 + 1$ and $X^\alpha = H^{4(\alpha-\frac{1}{2})}(\mathbb{R}^N)$ (see [13, Corollary 2.7] for the action of the semigroup generated by $-\Delta^2 - 1$ on the scale of Bessel potential spaces). Then [12, (2.2.3)] gives the result with $\beta = 4(\alpha - \frac{1}{2})$ for any $\alpha < 1$ close enough to 1. Applying (2.6), we see that (3.1) actually holds for any $\beta < 2$. □

We are ready to prove Theorems 1.1 and 1.3 announced in the Introduction.

Proof of Theorems 1.1 and 1.3. The quasi-stability of the semigroup $\{S(t)\}_{t\ge0}$ in $H^1(\mathbb{R}^N)$ generated by (1.1)–(1.2), which has already been proved in Theorem 1.2, guarantees that the assumptions of Theorem A.4 are fulfilled with $V = S(t_*)$. In particular, invoking (2.17) with constants $t_* \ge 1$, $q_* \in (0,1)$ and $\xi_* > 0$ and the global Lipschitz condition (2.19) for $K \colon \mathbf{B} \to Z$ with a Lipschitz constant $\kappa_* > 0$, it follows that for any $\theta \in (0, 1-q_*)$ there exists a compact set $\mathbf{E} \subset \mathbf{B}$ in $H^1(\mathbb{R}^N)$ such that $S(t_*)\mathbf{E} \subset \mathbf{E}$ and for some $c > 0$

$$d(S(kt_*)\mathbf{B}, \mathbf{E}) \le c(q_* + \theta)^k,\ k \in \mathbb{N}, \tag{3.2}$$

and

$$\dim_f(\mathbf{E}) \le \log_{\frac{1}{q_*+\theta}} \mathfrak{m}_{\frac{2\kappa_*}{\theta}}, \tag{3.3}$$

where $\mathfrak{m}_{\frac{2\kappa_*}{\theta}}$ is the maximal number of points z_j in a closed ball $\{z \in Z \colon \|z\|_Z \le \frac{2\kappa_*}{\theta}\}$ in Z with the property that $\xi_* \|z_j - z_l\|_{L^2(0,t_*;L^{2r'}_\varphi(\mathbb{R}^N))} > 1$ whenever $j \neq l$. Since by invariance of $\mathbf{A}$ we have $\mathbf{A} = S(kt_*)\mathbf{A} \subset S(kt_*)\mathbf{B}$, we see that $\mathbf{A} \subset \mathbf{E}$ and thus the global attractor $\mathbf{A}$ has its fractal dimension estimated as in (1.19).

Setting

$$\mathbf{M} = \bigcup_{t\in[t_*,2t_*]} S(t)\mathbf{E} \subset \mathbf{B},$$

it follows that $\mathbf{M}$ is compact in $H^1(\mathbb{R}^N)$, since $S(t)u_0$ is continuous in $H^1(\mathbb{R}^N)$ with respect to a pair of arguments $(t, u_0) \in [0,\infty) \times H^1(\mathbb{R}^N)$. Moreover, $S(t)\mathbf{M} \subset \mathbf{M}$ for $t \ge 0$, because $S(t_*)\mathbf{E} \subset \mathbf{E}$. Using Lemma 2.3 with $\alpha = \frac{1}{4}$ and (3.1) with $\beta = 1$, the fractal dimension of $\mathbf{M}$ in $H^1(\mathbb{R}^N)$ is estimated from above by $\frac{1}{\zeta}(1 + \dim_f(\mathbf{E}))$ (see [17, Proposition 3.1.13]). Since this holds for any $\zeta \in (0,1)$, we get from (3.3)

$$\dim_f(\mathbf{M}) \le 1 + \log_{\frac{1}{q_*+\theta}} \mathfrak{m}_{\frac{2\kappa_*}{\theta}}.$$

Note that if B is bounded in $H^1(\mathbb{R}^N)$, then there exists $t_B \ge 0$ such that $S(t_B)B \subset \mathbf{B}$. For $t \ge t_B + 2t_*$, we have $t - t_B = kt_* + t_* + r$ with some $r \in [0, t_*]$, $k \in \mathbb{N}$, and using again Lemma 2.3 with $\alpha = \frac{1}{4}$, we obtain from (3.2)

$$\begin{aligned} d(S(t)B, \mathbf{M}) &= d(S(t-t_B)S(t_B)B, \mathbf{M}) \le d(S(t_*+r)S(kt_*)\mathbf{B}, S(t_*+r)\mathbf{E}) \\ &\le \mu d(S(kt_*)\mathbf{B}, \mathbf{E}) \le \mu c (q_*+\theta)^k \le C_B e^{-\xi t}, \end{aligned}$$

where $\xi = -\frac{1}{t_*}\ln(q_*+\theta) > 0$ and $C_B > 0$. Hence $\mathbf{M}$ enjoys the properties (i), (ii) and (iii) listed in the Introduction and thus is an exponential attractor for the semigroup $\{S(t)\}_{t\ge0}$ with the upper bound of its fractal dimension as in (1.20).

If $0 < q_* < \frac{1}{2}$ we moreover apply [19, Theorem 2.1, Corollary 3.5]. Based on this we obtain, in turn, the estimate of fractal dimension of the global attractor $\mathbf{A}$ expressed through the Kolmogorov $\varepsilon-$entropy as in (1.21). Also, $\mathbf{A}$ is then contained in an exponential attractor $\mathbf{M}$ satisfying the estimate (1.22). □

Appendix A. Estimate of fractal dimension via method of quasi-stability

We recall that given in a Banach space X a compact set $\mathscr{B}$ its *fractal dimension* is

$$\dim_f(\mathscr{B}) = \limsup_{\varepsilon\to0} \log_{\frac{1}{\varepsilon}} n_\varepsilon,$$

where n_ε denotes the smallest number of ε–balls needed to cover $\mathscr{B}$ in X. In what follows we include key notions and results connected with estimates of fractal dimension using the method of quasi-stability (see e.g. [18, 17]).

Definition A.1 ([17, Definition 3.1.14])**.** We say that $n_Z \colon Z \to [0,\infty)$ is a *compact seminorm* on a normed space Z if and only if n_Z satisfies

$$n_Z(x+y) \le n_Z(x) + n_Z(y), \quad n_Z(\lambda x) = |\lambda|\, n_Z(x),\ x, y \in Z,\ \lambda \in \mathbb{R},$$

and each sequence $\{z_n\}$ bounded in Z contains a subsequence $\{z_{n_j}\}$ such that

$$n_Z(z_{n_j} - z_{n_l}) \to 0 \ \text{ as } \ j, l \to \infty.$$

Definition A.2 ([17, Definition 3.4.1])**.** Let X be a Banach space and $\mathscr{B}$ be a subset of X. We say that a mapping $V\colon \mathscr{B} \to X$ is *quasi-stable on* $\mathscr{B}$ if there is a normed space Z with a compact seminorm n_Z on Z, and a mapping $K\colon \mathscr{B} \to Z$ such that for some constants $q_* \in [0,1)$, $\kappa_* > 0$ we have

$$\|Kx - Ky\|_Z \le \kappa_* \|x-y\|_X\,, \; x,y \in \mathscr{B}, \tag{A.1}$$

$$\|Vx - Vy\|_X \le q_* \|x-y\|_X + n_Z(Kx - Ky), \; x,y \in \mathscr{B}. \tag{A.2}$$

In case of a semigroup $\{S(t)\}_{t\ge 0}$ on X, we say that the *dynamical system* $(X, S(t))$ *is quasi-stable on* $\mathscr{B} \subset X$ *at time* $t_* > 0$ if the mapping $S(t_*)$ is quasi-stable on $\mathscr{B}$.

Theorem A.3 ([17, Theorem 3.1.21])**.** *Let X be a Banach space, $\mathscr{B}$ be bounded and closed in X and $V\colon \mathscr{B} \to X$ be quasi-stable on $\mathscr{B}$ as in Definition A.2. If $\mathscr{B} \subset V\mathscr{B}$ then $\mathscr{B}$ is actually compact in X, its fractal dimension $\dim_f(\mathscr{B})$ in X is finite and, given any $\theta \in (0, 1-q_*)$,*

$$\dim_f(\mathscr{B}) \le \log_{\frac{1}{q_*+\theta}} \mathfrak{m}_{\frac{2\kappa_*}{\theta}}\,,$$

where $\mathfrak{m}_{\frac{2\kappa_*}{\theta}}$ is the maximal number of points z_j in the ball $\{z \in Z\colon \|z\|_Z \le \frac{2\kappa_*}{\theta}\}$ such that $n_Z(z_j - z_l) > 1$ whenever $j \ne l$.

Theorem A.4 ([17, Theorem 3.2.1])**.** *Let X be a Banach space, $\mathscr{B}$ be bounded and closed in X and $V\colon \mathscr{B} \to X$ be quasi-stable on $\mathscr{B}$ as in Definition A.2. If $V\mathscr{B} \subset \mathscr{B}$ then given $\theta \in (0, 1-q_*)$, there exists a compact set $\mathscr{E}_\theta \subset \mathscr{B}$ satisfying*

$$V\mathscr{E}_\theta \subset \mathscr{E}_\theta, \qquad d(V^k\mathscr{B}, \mathscr{E}_\theta) \le r(q_*+\theta)^k, \; k \in \mathbb{N},$$

for some $r > 0$, and also such that its fractal dimension in X is finite and estimated by

$$\dim_f(\mathscr{E}_\theta) \le \log_{\frac{1}{q_*+\theta}} \mathfrak{m}_{\frac{2\kappa_*}{\theta}}\,,$$

where $\mathfrak{m}_{\frac{2\kappa_*}{\theta}}$ is as in Theorem A.3.

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