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Evolution Equations with Sectorial Operator on Fractional Power Scales

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Abstract
Originating with the famous monograph by Dan Henry, the semigroup approach to evolution problems having a positive sectorial operator in the main part allows us to settle them at various levels of the fractional power scale associated with the main linear operator. This translates into different regularity properties of local solutions to such equations. Specific applications of the abstract results to the 2D surface quasi-geostrophic equation or the fractional chemotaxis system are presented.

Keywords Sectorial operators · Fractional scale of Banach spaces · Semilinear sectorial equations · Local and global solutions · Quasi-geostrophic equation · Fractional chemotaxis system

Mathematics Subject Classification Primary 35K90; Secondary 35B65 · 35S15 · 35Q35

1 Introduction

This paper is devoted to the intensively studied problems of regularity and global in time continuation of the local solutions to semilinear evolution equations. Our approach follows closely the semigroup approach known from the classical monographs by Henry [24] and Pazy [36] and makes use of the important results of Triebel [43] and Amann [1] concerning the construction of the fractional power scales connected with positive operators. Our aim here is to discuss applications of these abstract results for semilinear equations with sectorial positive operator in the main part to a class of new examples.

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The paper consists of two parts. In the first one we recall known from the literature abstract facts needed in further considerations. A variety of formulas and properties of fractional powers of positive operators and semigroups generated by sectorial operators are first recalled, with detailed presentation to be found e.g. in [1, 6, 14, 24, 25, 33]. Next we list basic properties of the fractional power spaces, in which the semilinear equations will be studied. In the literature there are known a few papers and monographs dealing with fractional power scales and equations on it. From among of them we select here, [1, 5, 8, 16, 23, 26, 29, 31, 38, 39]. Not only do we discuss in Sect. 2 the one-sided scale associated to non-negative fractional powers of positive operators, but also extend the scale to the negative side by extrapolating the main positive sectorial operator in a reflexive Banach space. This allows us to consider evolution equations in the extrapolated fractional power scale, which in applications may include spaces of continuous functionals on subspaces of classical Sobolev spaces. A brief recapitulation of the local solvability result by Henry [24] (later extended in [6]) in the new context is also mentioned. The abstract technique used in this paper is based on locating the considered problem at an admissible level of the fractional power scale associated with the sectorial or elliptic operator from the main part. Once the level is set, it provides the *base space* in which we need to place the action of the nonlinearity, specific in a particular example. We need to assure that the nonlinearity acts between suitably chosen levels of that fractional power scale; more precisely, it is Lipschitz continuous on bounded sets as a map between elements of the scale—the *phase space* and the *base space*. Then our original studies are reported starting with the global extendibility for small data solutions in Lemma 2.28.

The second part of the paper is devoted to two examples known from the recent references [3, 10, 12, 44]—the *surface quasi-geostrophic equation* and a version of the *chemotaxis system*. Using presented earlier abstract results and constructing suitable estimates, we discuss a possible extent of regularity of the local solutions in these examples. The problems studied here are subject to Dirichlet boundary conditions in bounded regular domains, which is different from examples of Cauchy problems in the whole of $\mathbb{R}^N$ treated in some of the above-mentioned references, where the complications connected with boundary conditions are absent. Very helpful in these studies is the idea introduced by Giga and Miyakawa [22] in case of the celebrated Navier–Stokes equation. Thus, in some sense, we continue here the direction used in [7], where the results of [22] were used. We also give an application of Lemma 2.28 on small data solutions to the studied examples.

In Sect. 3 we consider the abstract Cauchy problem

$$\theta_t + A\theta = F(\theta), \quad \theta(0) = \theta_0, \tag{1.1}$$

associated with the 2D quasi-geostrophic equation in a bounded C^2 domain $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} &\theta_t + u \cdot \nabla\theta + \kappa(-\Delta)^{\omega}\theta = f, \quad x \in \Omega, \quad t > 0, \\ &\theta = 0 \quad \text{on } \partial\Omega, \quad \theta(0, x) = \theta_0(x), \quad x \in \Omega, \end{aligned}$$

with a parameter $\omega \in \left(\frac{1}{2}, 1\right]$, where the velocity field $u = (u_1, u_2)$ is expressed by the stream function ψ through the relation $u = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}\right)$, where $(-\Delta)^{\frac{1}{2}}\psi = -\theta$. Here θ represents the potential temperature, $\kappa > 0$ is a diffusivity coefficient, f is a free time-independent force.

Considering (1.1) in the one-sided fractional power scale generated by $A = (-\Delta)^{\omega}$ in $X^0 = L^2(\Omega)$, we show in Theorem 3.4 that unique local (X^{α}, X^0) solutions of (1.1) exist for $\alpha \in \left(\frac{1}{2\omega}, 1\right)$ and $f \in L^2(\Omega)$. Knowing that $f \in L^{\infty}(\Omega)$, they are in fact global in time and bounded as shown in Proposition 3.5. Smoother solutions of 2D quasi-geostrophic equation can be obtained by considering (1.1) in the base space from the positive part of the one-sided scale. In Theorem 3.6 we show that if $f \in X^{\beta} = D((-\Delta)^{\omega\beta}), 0 < \beta < \frac{1}{4\omega}$, and $\alpha \in \left[\beta + \frac{1}{2\omega}, \beta + 1\right)$, then local (X^{α}, X^{β}) solutions of (1.1) exist. We use Lemma 2.28 in order to determine in Proposition 3.8 the range of parameters for which these solutions are in fact global in time, provided that the initial data is sufficiently small.

In Sect. 3.3, borrowing the idea from the paper [22] on the Navier–Stokes equation, we consider (1.1) in the spaces of negative part of the extrapolated fractional scale generated by A. We prove in Theorem 3.11 that if $f \in X^{\beta}$ with $\beta \in \left(\frac{1}{\omega} - 2, 0\right)$ and $\alpha \in \left[\frac{1}{2}\beta + \frac{1}{2\omega}, \beta + 1\right)$, then for each $\theta_0 \in X^{\alpha}$ there exists a unique (X^{α}, X^{β}) solution of (1.1). Invoking the natural $L^{\infty}(\Omega)$ a priori estimate for the 2D quasi-geostrophic equation, in Proposition 3.14 we show that these (X^{α}, X^{β}) solutions of (1.1) are in fact global in time for $f \in L^{\infty}(\Omega)$, $\beta \in \left(\frac{1}{2\omega} - 1, 0\right]$ and $\frac{1}{2\omega} < \alpha < \beta + 1$.

A similar analysis is made in Sect. 4 devoted to the abstract Cauchy problem

$$u_t + Au = F(u), \quad u(0) = u_0, \tag{1.2}$$

related to the fractional chemotaxis system in a bounded C^2 domain Ω of $\mathbb{R}^N$

$$\begin{aligned} &u_t + (-\Delta)^{\omega}u + \nabla \cdot (u\nabla v) = 0, \quad x \in \Omega,\ t > 0,\\ &\Delta v + u = 0, \quad x \in \Omega, \quad t > 0,\\ &u = v = 0 \ \text{ on } \ \partial\Omega,\\ &u(0, x) = u_0(x), \quad x \in \Omega, \end{aligned}$$

with a parameter $\omega \in \left(\frac{1}{2}, 1\right]$. Local (X^{α}, X^0) solutions of (1.2) with $\alpha \in \left[\frac{1}{2\omega}, 1\right)$ are obtained in Theorem 4.1 and Remark 4.2 in $X^0 = L^p(\Omega)$, $p > N$, and $X^0 = L^2(\Omega)$ with $N = 3$, respectively. The latter ones are global in time for small initial conditions, see Corollary 4.4. The corresponding result on local solvability of fractional chemotaxis equation in smoother spaces is given in Theorem 4.5. Finally, the problem (1.2) considered on the extrapolated part of the fractional powers scale generated by $(-\Delta)^{\omega}$, $\omega \in \left(\frac{1}{2}, 1\right]$, in $L^2(\Omega)$ with $N \geq 2$ is shown in Theorem 4.7 to possess local (X^{α}, X^{β}) solutions provided that $\beta \in (-1, 0]$ and $\alpha > 0$ are such that $2\alpha - \beta \geq \frac{N}{4\omega}$ and $\frac{1}{2\omega} < \alpha - \beta < 1$.

There are two notational conventions used throughout the text. For a real number s by s_- we understand a number strictly smaller but eventually close to s. For convenience we also use the letter c to denote a positive constant that can differ even within one calculation.

2 Scales of Banach Spaces Associated with Positive Operators

A partial differential equation which can be written in the semilinear form

$$u_t + \mathcal{A}u = \mathcal{F}(u) \tag{2.1}$$

has various interpretations depending on the choice of the base space in which the realization of the linear differential operator $\mathcal{A}$ is understood. This choice is dictated mainly by the boundary conditions associated with the equation, if any, by the properties of the realization we want to exploit in the arguments, and by the suitable interplay between the linear part and the qualities of the nonlinear part $\mathcal{F}$ with respect to the subspaces of the base space. All these features eventually become determined in a suitable definition of a solution in order to prove its existence for a sufficiently large set of initial conditions and to study its long time behavior as time evolves. One of the successful approaches to treat large classes of partial differential equations emerged forty years ago in the monograph of Henry [24] where a given realization of the linear operator $\mathcal{A}$ in (2.1) in a chosen Banach space was a sectorial operator.

Definition 2.1 Let $\phi \in \big(0, \frac{\pi}{2}\big)$, $M \geq 1$ and $a \in \mathbb{R}$. We say that a densely defined closed linear operator $A\colon X \supseteq \mathrm{dom}(A) \to X$ in a Banach space X is *sectorial (of type* (a, ϕ, M)*)* if the sector $S_{a,\phi} = \{\lambda \in \mathbb{C}\colon \phi \leq |\arg(\lambda - a)| \leq \pi,\ \lambda \neq a\}$ is contained in the resolvent set $\rho(A)$ and the resolvent operator $R(\lambda\colon A) = (\lambda - A)^{-1}$ is estimated by $\|R(\lambda\colon A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - a|}$ for $\lambda \in S_{a,\phi}$. We say that a sectorial operator A is *positive sectorial* if $\mathrm{Re}\,\sigma(A) > 0$, where $\sigma(A)$ denotes the spectrum of the operator A.

The well-known result from the semigroup theory is that sectorial operators are precisely the negative generators of analytic semigroups.

Definition 2.2 Let X be a Banach space and $\{T(t)\colon t \geq 0\}$ be a C^0 semigroup in X. We say that $\{T(t)\colon t \geq 0\}$ is an *analytic semigroup* if there exists a family of linear bounded operators $T(z)$ in X defined for z from a sector $\Delta_\phi = \{z \in \mathbb{C}\colon |\arg z| < \phi\}$ for some $0 < \phi \leq \frac{\pi}{2}$, which coincide with $T(t)$ for $t \geq 0$, and such that the mapping $z \mapsto T(z)$ is analytic in $\Delta_\phi \setminus \{0\}$, for each $v \in X$ we have $T(z)v \to v$ as $z \to 0$, and $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta_\phi$.

The following theorem holds (see e.g. [24, Theorem 1.3.4], [14, Theorem 2.2.7]).

Theorem 2.3 *A linear operator $A\colon X \supseteq \mathrm{dom}(A) \to X$ in a Banach space X is a negative generator of an analytic semigroup $\{e^{-At}\colon t \geq 0\}$ of bounded linear operators $e^{-At}\colon X \to X$, $t \geq 0$, if and only if A is a sectorial operator in X. If A is sectorial of type (a, ϕ, M) in X, then the semigroup is defined via a Dunford integral*

$$e^{-At} = -\frac{1}{2\pi i}\int_\Phi e^{-\lambda t} R(\lambda\colon A)d\lambda \in \mathcal{L}(X), \quad t > 0,$$

where Φ *is a path from* $a + \infty e^{-i\theta}$ *to* $a + \infty e^{i\theta}$ *for* $\theta \in \left[\phi, \frac{\pi}{2}\right)$ *and surrounding* a *in order to avoid* $[a, \infty)$. *If* $\operatorname{Re}\sigma(A) > a$ *then for some* $C > 0$ *we have*

$$\left\|e^{-At}\right\|_{\mathcal{L}(X)} \leq Ce^{-at}, \quad \left\|Ae^{-At}\right\|_{\mathcal{L}(X)} \leq \frac{C}{t}e^{-at}, \quad t > 0.$$

2.1 Fractional Powers

Positive sectorial operators are in particular positive operators for which fractional powers can be defined.

Definition 2.4 We say that $A\colon X \supseteq \operatorname{dom}(A) \to X$ is a *positive operator* in a Banach space X (or *of positive type* K in the nomenclature of Amann [1, Sect. III.4.6]) if A is a densely defined closed linear operator, $(-\infty, 0] \subseteq \rho(A)$ and $\|R(s\colon A)\|_{\mathcal{L}(X)} \leq \frac{K}{1+|s|}$, $s \leq 0$ with some $K \geq 1$.

In fact, if A is a positive operator, then there exist $0 < \varepsilon < \pi$, $r > 0$ and $K_1 \geq 1$ such that $S_{0,\varepsilon} = \{\lambda \in \mathbb{C}\colon \varepsilon \leq |\arg\lambda| \leq \pi, \lambda \neq 0\} \subseteq \rho(A)$, $\overline{B}(0, r) \subseteq \rho(A)$, and

$$\|R(\lambda\colon A)\|_{\mathcal{L}(X)} \leq \frac{K_1}{1 + |\lambda|} \quad \text{for } \lambda \in S_{0,\varepsilon} \cup \overline{B}(0, r). \tag{2.2}$$

Definition 2.5 ([1, Sect. I.2.9]) Assume that A is a positive operator in a Banach space X and (2.2) holds. Then for $z \in \mathbb{C}$ such that $\operatorname{Re} z > 0$ a bounded linear operator A^{-z} is defined via a Dunford integral

$$A^{-z} = -\frac{1}{2\pi i}\int_{\Phi} \lambda^{-z}R(\lambda\colon A)d\lambda = \frac{1}{2\pi i}\int_{\Phi'} (-\mu)^{-z}(\mu + A)^{-1}d\mu \in \mathcal{L}(X),$$

where Φ is a path from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for $\theta \in [\varepsilon, \pi)$ and surrounding 0 in order to avoid $(-\infty, 0]$, whereas Φ' is a path from $\infty e^{-i(\pi-\theta)}$ to $\infty e^{i(\pi-\theta)}$ avoiding $[0, \infty)$. For $z = 0$, A^0 is defined as an identity operator I on X.

Using Euler's Γ function, these fractional powers can be expressed as

$$A^{-z} = \frac{\Gamma(n)}{\Gamma(n - z)\Gamma(z)}\int_0^{\infty} s^{-z+n-1}[(s + A)^{-1}]^n ds \tag{2.3}$$

for a given $n \in \mathbb{N}$ and $z \in \mathbb{C}$ such that $0 < \operatorname{Re} z < n$, see [14, Theorem 3.0.7].

The operator A^{-z} with $\operatorname{Re} z > 0$ is invertible from X onto $\operatorname{im}(A^{-z})$ which allows to extend the above definition.

Definition 2.6 For $z \in \mathbb{C}$ such that $\operatorname{Re} z > 0$ we define the operator

$$A^z := (A^{-z})^{-1}\colon X \supseteq \operatorname{dom}(A^z) = \operatorname{im}(A^{-z}) \to X.$$

The fractional powers of positive operators enjoy many useful properties including the following.

Proposition 2.7 *A^z with $\operatorname{Re} z > 0$ is a densely defined closed linear operator and for $n \in \mathbb{N}$ the operator A^n coincides with the nth iterate of A. If $z_1, z_2 \in \mathbb{C}$ are such that $\operatorname{Re} z_1 > \operatorname{Re} z_2 > 0$, then $\operatorname{dom}(A^{z_1}) \subseteq \operatorname{dom}(A^{z_2}) \subseteq X$. For $\alpha \in \mathbb{R}$ it follows that*

$$A^\alpha R(\lambda\colon A)x = R(\lambda\colon A)A^\alpha x, \quad x \in \operatorname{dom}(A^\alpha), \quad \lambda \in \rho(A). \tag{2.4}$$

For $\alpha, \beta \in \mathbb{R}$ we also have

$$A^{\alpha+\beta}x = A^\alpha A^\beta x = A^\beta A^\alpha x, \quad x \in \operatorname{dom}(A^\mu), \quad \mu = \max\{\alpha, \beta, \alpha+\beta\}.$$

If $-\infty < \gamma \le \alpha < \infty$ and $x \in \operatorname{dom}(A^\alpha)$, then $x \in \operatorname{dom}(A^\gamma)$ and $A^\gamma x \in \operatorname{dom}(A^{\alpha-\gamma})$ and $A^{\alpha-\gamma}A^\gamma x = A^\alpha x$.

In the vein of (2.3), the fractional powers of exponents with positive real part can be alternatively expressed as

$$A^z x = \frac{\Gamma(n)}{\Gamma(n-z)\Gamma(z)} \int_0^\infty s^{z-1}[A(s+A)^{-1}]^n x\,ds, \quad x \in \operatorname{dom}(A^n) \tag{2.5}$$

for $0 < \operatorname{Re} z < n$ with $n \in \mathbb{N}$. Since $\Gamma(1) = 1$, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$ and

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \text{ for } \operatorname{Re} z > 0, \quad z \notin \mathbb{N}, \quad \Gamma(z+1) = z\Gamma(z) \quad \text{for } \operatorname{Re} z > 0,$$

we derive from (2.5) the *Balakrishnan formulas* (see [33, (3.1)], [25]), that is, for $n \in \mathbb{N}_0$ and $0 < \operatorname{Re} z < 1$ we have

$$A^{n+z}x = \frac{n!\sin(\pi z)}{\pi z(z+1)\cdots(z+n-1)} \int_0^\infty s^{n-1+z}[A(s+A)^{-1}]^{n+1} x\,ds, \quad x \in \operatorname{dom}(A^{n+1}).$$

Note that by (2.4) in the above formula we can write

$$[A(s+A)^{-1}]^{n+1}x = [(s+A)^{-1}]^{n+1}A^{n+1}x \quad \text{for } x \in \operatorname{dom}(A^{n+1}), \quad n \in \mathbb{N}_0.$$

Lemma 2.8 ([1, (V.1.2.12)]) *Let $-\infty < \gamma < \beta < \alpha < \infty$ and $A\colon X \supseteq \operatorname{dom}(A) \to X$ be a positive operator in a Banach space X. Then the* moment inequality *holds, that is,*

$$\left\|A^\beta x\right\|_X \le c \left\|A^\gamma x\right\|_X^{\frac{\alpha-\beta}{\alpha-\gamma}} \left\|A^\alpha x\right\|_X^{\frac{\beta-\gamma}{\alpha-\gamma}}, \quad x \in \operatorname{dom}(A^\alpha) \tag{2.6}$$

with some constant $c = c(\alpha, \beta, \gamma) > 0$.

If A is a positive operator in a reflexive Banach space X, then its adjoint A^* is also a positive operator, since they have the same resolvent set and resolvent estimate and in (2.2) one can change A into A^* and X into X^* with the same ε, K_1 and r.

Lemma 2.9 ([1, Lemma V.1.4.11]) *If $A\colon X \supseteq \operatorname{dom}(A) \to X$ is a positive operator in a reflexive Banach space X, then*

$$(A^\alpha)^* = (A^*)^\alpha \quad \textit{for } \alpha \in \mathbb{R}.$$

Positive sectorial operators are examples of positive operators and their fractional powers can be expressed via the analytic semigroups generated by their negatives (cp. [1, Theorem III.4.6.6]).

Proposition 2.10 *If A is a positive sectorial operator in a Banach space X, then for* $\operatorname{Re} z > 0$ *we have*

$$A^{-z}x = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-At} x dt, \quad x \in X.$$

Given any $\gamma > 0$, $e^{-At} \colon X \to \operatorname{dom}(A^\gamma)$, $A^\gamma e^{-At} x = e^{-At} A^\gamma x$ for $x \in \operatorname{dom}(A^\gamma)$ and there exist $a > 0$ and $C_\gamma > 0$ such that

$$\left\| e^{-At} \right\|_{\mathcal{L}(X)} \leq Ce^{-at}, \quad \left\| A^\gamma e^{-At} \right\|_{\mathcal{L}(X)} \leq \frac{C_\gamma}{t^\gamma} e^{-at}, \quad t > 0. \tag{2.7}$$

As observed in [1, Proposition I.1.2.3], [36, Corollary 1.10.6], if A is a (positive) sectorial operator in a reflexive Banach space X, then A^* is also a (positive) sectorial operator in X^* and $(e^{-At})^* = e^{-A^*t}$, $t \geq 0$.

2.2 One-Sided Fractional Power Scale

Having recalled the main properties of the fractional powers of positive operators, we now observe that they define a scale of Banach spaces and give rise to a scale of positive operators.

Definition 2.11 Let $A \colon X \supseteq \operatorname{dom}(A) \to X$ be a positive operator in a Banach space X. For any $\alpha \geq 0$ we consider its fractional power A^α, which is a densely defined closed linear operator with $0 \in \rho(A^\alpha)$; $A^0 = I$. We define the *fractional power space* X_A^α as $\operatorname{dom}(A^\alpha)$ endowed with the equivalent graph norm of A^α, that is,

$$\|x\|_{\alpha,A} = \left\| A^\alpha x \right\|_X, \quad x \in X_A^\alpha = \operatorname{dom}(A^\alpha). \tag{2.8}$$

The space X_A^α is a Banach space and a dense subspace of X. If $0 \leq \alpha < \beta$ then X_A^β is densely and continuously injected into X_A^α, i.e., $i_{\beta,\alpha} \colon X_A^\beta \overset{d}{\hookrightarrow} X_A^\alpha$.

As another consequence of the above definition, the linear operator A^α restricted to X_A^β, $0 \leq \alpha \leq \beta$, or, in other words, A^α composed with the injection $X_A^\beta \hookrightarrow X_A^\alpha$, defines an isometry between elements of the scale.

Lemma 2.12 ([43, Sect. 1.15.2]) *For $0 \leq \alpha \leq \beta$ the operator*

$$A^\alpha|_{X_A^\beta} \colon X_A^\beta \to X_A^{\beta-\alpha} \tag{2.9}$$

is an isometry between Banach spaces, i.e., $A^\alpha|_{X_A^\beta} \in \mathcal{L}iso(X_A^\beta, X_A^{\beta-\alpha})$.

By the moment inequality (Lemma 2.8) we obtain a relation for the norms of fractional power spaces:

$$\|x\|_{\beta,A} \leq c\,\|x\|_{\gamma,A}^{\frac{\alpha-\beta}{\alpha-\gamma}}\,\|x\|_{\alpha,A}^{\frac{\beta-\gamma}{\alpha-\gamma}}, \quad x \in X_A^\alpha, \quad \text{whenever } 0 \leq \gamma < \beta < \alpha.$$

Along with the fractional power spaces, corresponding operators are defined as realizations of A in X_A^α.

Definition 2.13 For $\alpha \geq 0$ by $A_\alpha\colon X_A^\alpha \supseteq X_A^{\alpha+1} \to X_A^\alpha$ we denote the X_A^α-realization of A (via the injection $i_{\alpha,0}\colon X_A^\alpha \overset{d}{\hookrightarrow} X$) given by

$$A_\alpha x = Ax,\ x \in X_A^{\alpha+1} = \{x \in X_A^\alpha \cap X_A^1\colon Ax \in X_A^\alpha\}.$$

Thus A_α is a restriction of A to $X_A^{\alpha+1}$.

Note that $A_0 = A$ and $A^\alpha A_\alpha x = A^{\alpha+1}x$ for $x \in X_A^{\alpha+1}$ and $A_\alpha \in \mathcal{L}iso(X_A^{\alpha+1}, X_A^\alpha)$, since

$$\|A_\alpha x\|_{\alpha,A} = \left\|A^\alpha A_\alpha x\right\|_X = \left\|A^{\alpha+1}x\right\|_X = \|x\|_{\alpha+1,A}, \quad x \in X_A^{\alpha+1}.$$

Thus we get for $0 \leq \alpha \leq \beta$

$$A_\alpha \circ i_{\beta+1,\alpha+1}x = Ax = i_{\beta,\alpha} \circ A_\beta x, \quad x \in X_A^{\beta+1},$$

so that the diagram

$$\begin{array}{ccc} X_A^{\beta+1} & \overset{i_{\beta+1,\alpha+1}}{\longrightarrow} & X_A^{\alpha+1} \\ \big\downarrow A_\beta & & \big\downarrow A_\alpha \\ X_A^\beta & \underset{i_{\beta,\alpha}}{\longrightarrow} & X_A^\alpha \end{array}$$

is commutative. Observe that by (2.4) we also have

$$R(\lambda\colon A)X_A^\alpha \subseteq X_A^\alpha \quad \text{for } \lambda \in \rho(A),$$

which leads to the following result.

Lemma 2.14 *For A as in Definition 2.11, A_α is a densely defined closed operator in X^α and $0 \in \rho(A_\alpha)$. Moreover, $\rho(A) \subseteq \rho(A_\alpha)$ and $R(\lambda\colon A_\alpha) = R(\lambda\colon A)|_{X_A^\alpha}$ for $\lambda \in \rho(A)$. Furthermore, we have*

$$\|R(\lambda\colon A_\alpha)\|_{\mathcal{L}(X_A^\alpha)} \leq \|R(\lambda\colon A)\|_{\mathcal{L}(X)}, \quad \lambda \in \rho(A).$$

In particular, A_α is a positive operator (of the same type as A).

If $\alpha \in [0,1]$, then $\rho(A) = \rho(A_\alpha)$, since $\mathrm{dom}(A) \subseteq X_A^\alpha$, but in fact this can be extended to all $\alpha \geq 0$.

Lemma 2.15 *If $0 \le \alpha \le \beta$ then A_β is the X_A^β-realization of A_α (via $i_{\beta,\alpha}\colon X_A^\beta \hookrightarrow X_A^\alpha$). Thus if $\beta \in [\alpha, \alpha+1]$ then $\rho(A_\beta) = \rho(A_\alpha)$. Hence $\rho(A) = \rho(A_\alpha)$ for any $\alpha \ge 0$.*

Lastly, we have a description of fractional power spaces of operators A_α.

Lemma 2.16 ([1, Proposition V.1.2.6]) *For $\alpha, \gamma \ge 0$ we have*

$$\mathrm{dom}((A_\alpha)^\gamma) = X_A^{\alpha+\gamma} \quad \textit{and} \quad (A_\alpha)^\gamma x = A^\gamma x, \quad x \in X_A^{\alpha+\gamma}.$$

In particular,

$$(A_\alpha)^\gamma = A^\gamma|_{X_A^{\alpha+\gamma}} \in \mathcal{L}iso(X_A^{\alpha+\gamma}, X_A^\alpha) \quad \textit{and} \quad \left\|(A_\alpha)^\gamma x\right\|_{\alpha,A} = \|x\|_{\alpha+\gamma,A}\,, \quad x \in X_A^{\alpha+\gamma}. \tag{2.10}$$

Definition 2.17 Let $A\colon X \supseteq \mathrm{dom}(A) \to X$ be a positive operator in a Banach space X. Then the family of triples $(X_A^\alpha, \|\cdot\|_{\alpha,A}\,, A_\alpha)_{\alpha\ge 0}$ constructed above is called a *one-sided fractional power scale generated by A*. Hence $(X_A^\alpha, \|\cdot\|_{\alpha,A})$, $\alpha \ge 0$, is a family of Banach spaces and $A_\alpha \in \mathcal{L}iso(X_A^{\alpha+1}, X_A^\alpha)$, $\alpha \ge 0$, is a family of isometries from $X_A^{\alpha+1}$ onto X_A^α such that

(i) $i_{\beta,\alpha}\colon X_A^\beta \overset{d}{\hookrightarrow} X_A^\alpha$ for $0 \le \alpha < \beta$,
(ii) $A_\alpha \circ i_{\beta+1,\alpha+1} = i_{\beta,\alpha} \circ A_\beta$, $0 \le \alpha < \beta$,
(iii) $(A_\alpha)^\gamma = A^\gamma|_{X_A^{\alpha+\gamma}} \in \mathcal{L}iso(X_A^{\alpha+\gamma}, X_A^\alpha)$ and $\|(A_\alpha)^\gamma x\|_{\alpha,A} = \|x\|_{\alpha+\gamma,A}\,,\ x \in X_A^{\alpha+\gamma}$ for $\alpha, \gamma \ge 0$,
(iv) for $0 \le \gamma < \beta < \alpha$ there exists $c > 0$ such that

$$\|x\|_{\beta,A} \le c\, \|x\|_{\gamma,A}^{\frac{\alpha-\beta}{\alpha-\gamma}}\, \|x\|_{\alpha,A}^{\frac{\beta-\gamma}{\alpha-\gamma}}\,, \quad x \in X_A^\alpha.$$

If no confusion arises, we drop the subscript A and denote the space by X^α and the norm by $\|\cdot\|_\alpha$ in the one-sided fractional power scale.

Lemma 2.18 *If A is positive sectorial, then, for a given $\alpha \ge 0$, its realization A_α in X_A^α is also positive sectorial. Moreover, $e^{-A_\alpha t} = e^{-At}|_{X_A^\alpha}$ and there exists $a > 0$ such that for $\gamma \ge 0$ we have with some $C_\gamma > 0$*

$$\left\|e^{-A_\alpha t}\right\|_{\mathcal{L}(X_A^\alpha, X_A^{\alpha+\gamma})} \le \frac{C_\gamma e^{-at}}{t^\gamma}, \quad t > 0.$$

A one-sided fractional power scale can be generated by the realization of a given positive operator in a fractional power space.

Example 2.19 ([1, Proposition V.1.2.6]) Let $A\colon X \supseteq \mathrm{dom}(A) \to X$ be a positive operator in a Banach space X and, for a given $\gamma \ge 0$, let $A_\gamma\colon X_A^\gamma \supseteq X_A^{\gamma+1} \to X_A^\gamma$ be its realization in X_A^γ. Then the one-sided fractional power scale generated by A_γ coincides with shifted by $\gamma \ge 0$ the one-sided fractional power scale generated by A, that is,

$$\left((X_A^\gamma)_{A_\gamma}^\alpha\,, \|\cdot\|_{\alpha,A_\gamma}\,, (A_\gamma)_\alpha\right)_{\alpha\ge 0} = \left(X_A^{\alpha+\gamma}, \|\cdot\|_{\alpha+\gamma,A}\,, A_{\alpha+\gamma}\right)_{\alpha\ge 0}.$$

A one-sided fractional power scale can be generated by the adjoint operator of a given positive operator in a reflexive Banach space.

Example 2.20 Let $A\colon X \supseteq \mathrm{dom}(A) \to X$ be a positive operator in a reflexive Banach space X and let $A^*\colon X^* \supseteq \mathrm{dom}(A^*) \to X^*$ denote its adjoint operator. Then $((X^*)^\alpha_{A^*}, \|\cdot\|_{\alpha,A^*}, (A^*)_\alpha)_{\alpha\geq 0}$ is a one-sided fractional power scale generated by A^*.

A one-sided fractional power scale can also be generated by the extrapolated operator of a given positive operator, which we describe below.

Let $A\colon X \supseteq \mathrm{dom}(A) \to X$ be a positive operator in a Banach space $(X, \|\cdot\|_X)$. Since $0 \in \rho(A)$, we have $A^{-1} \in \mathcal{L}(X)$ and $|x|_X = \left\|A^{-1}x\right\|_X$, $x \in X$, is another norm on X.

We consider a completion $(X^{-1}, \|\cdot\|_{-1})$ of the normed space $(X, |\cdot|_X)$ via an isometric embedding $j\colon (X, |\cdot|_X) \to (X^{-1}, \|\cdot\|_{-1})$. We call the Banach space X^{-1} an *extrapolated space*. Recall that $j(X)$ is a dense subspace of X^{-1} and

$$\|jx\|_{-1} = \left\|A^{-1}x\right\|_X, \quad x \in X \quad \text{and} \quad \|jAx\|_{-1} = \|x\|_X, \quad x \in \mathrm{dom}(A).$$

Therefore, using j we consider A with domain in X^{-1} and values in X^{-1}, that is, $A_{X^{-1}}\colon X^{-1} \supseteq j(\mathrm{dom}A) \to X^{-1}$ given by

$$A_{X^{-1}}y = jAj^{-1}y, \quad y \in j(\mathrm{dom}(A)) = \mathrm{dom}(A_{X^{-1}}).$$

Since $\mathrm{dom}(A)$ is dense in X, there exists a unique linear operator

$$A_{-1}\colon X^{-1} \supseteq j(X) = \mathrm{dom}(A_{-1}) \to X^{-1},$$

which extends $A_{X^{-1}}$ and such that $A_{-1} \circ j \in \mathcal{L}iso(X, X^{-1})$, i.e., $A_{-1} \circ j$ is an isometry from X (endowed with the original norm $\|\cdot\|_X$) onto $(X^{-1}, \|\cdot\|_{-1})$, so

$$\|A_{-1}jx\|_{-1} = \|x\|_X, \quad x \in X.$$

We call A_{-1} the *extrapolated operator* A in the extrapolated space X^{-1}. Thus we have

$$A_{-1}jx = A_{X^{-1}}jx = jAx, \quad x \in \mathrm{dom}(A),$$

so it is understood as a realization of the operator A in X^{-1} in this sense.

Lemma 2.21 *A_{-1} is a densely defined closed linear operator, A_{-1} is bijective, $\rho(A_{-1}) = \rho(A)$ and $R(\lambda\colon A) = j^{-1}R(\lambda\colon A_{-1})j$ for $\lambda \in \rho(A)$. In particular, we get $A_{-1}^{-1} \in \mathcal{L}(X^{-1})$ and*

$$A_{-1}^{-1}jx = jA^{-1}x \quad \text{for } x \in X.$$

We also have $\|R(\lambda\colon A_{-1})\|_{\mathcal{L}(X^{-1})} \leq \|R(\lambda\colon A)\|_{\mathcal{L}(X)}$ for $\lambda \in \rho(A)$. Therefore, A_{-1} is a positive operator (of the same type as A).

Using the definition of a fractional power of a positive operator and properties of the extrapolated operator, we obtain the following relations.

Lemma 2.22 *For* $\alpha \in \mathbb{R}$ *we have*

$$A_{-1}^{\alpha} jx = jA^{\alpha}x, \quad x \in \mathrm{dom}(A^{\alpha}),$$

and $\mathrm{dom}(A_{-1}^{\alpha+1}) = j(\mathrm{dom}(A^{\alpha}))$ *for* $\alpha \geq 0$.

Example 2.23 Let $A\colon X \supseteq \mathrm{dom}(A) \to X$ be a positive operator in a Banach space X and $(X^{-1}, \|\cdot\|_{-1})$ be a completion of the normed space $(X, \left\|A^{-1}x\right\|_X)$ via an isometric embedding $j\colon (X, \left\|A^{-1}x\right\|_X) \to (X^{-1}, \|\cdot\|_{-1})$. Let $A_{-1}\colon X^{-1} \supseteq j(X) \to X^{-1}$ be the corresponding extrapolated operator. Then $((X^{-1})_{A_{-1}}^{\alpha}, \|\cdot\|_{\alpha,A_{-1}}, (A_{-1})_{\alpha})_{\alpha\geq 0}$ is a one-sided fractional power scale generated by A_{-1}.

Note that in the above the choice of j determines X^{-1}. Without its specification X^{-1} is only determined up to an isometry. A particular choice of an isometric embedding j is made if $A\colon X \supseteq \mathrm{dom}(A) \to X$ is a positive operator in a *reflexive* Banach space X. Namely, we have one-sided fractional power scales $(X_A^{\alpha}, \|\cdot\|_{\alpha,A}, A_{\alpha})_{\alpha\geq 0}$ and $((X^*)_{A^*}^{\alpha}, \|\cdot\|_{\alpha,A^*}, (A^*)_{\alpha})_{\alpha\geq 0}$ generated by A and its adjoint operator A^*, respectively. Since X_A^{α} is isometric to X and $(X^*)_{A^*}^{\alpha}$ is isometric to X^*, each spaces X_A^{α} and $(X^*)_{A^*}^{\alpha}$ are reflexive Banach spaces. Moreover, we have dense and continuous injections $i_{\beta,\alpha}\colon X_A^{\beta} \overset{d}{\hookrightarrow} X_A^{\alpha}$ and $i_{*,\beta,\alpha}\colon (X^*)_{A^*}^{\beta} \overset{d}{\hookrightarrow} (X^*)_{A^*}^{\alpha}$ for $0 \leq \alpha < \beta$.

For $\alpha \geq 0$ we denote by $[(X^*)_{A^*}^{\alpha}]^*$ the space of all continuous functionals on $(X^*)_{A^*}^{\alpha}$, which is a reflexive Banach space. The adjoint map $(i_{*,\beta,\alpha})^* \in \mathcal{L}([(X^*)_{A^*}^{\alpha}]^*, [(X^*)_{A^*}^{\beta}]^*)$, $0 \leq \alpha < \beta$, given by

$$(i_{*,\beta,\alpha})^* x^{**} = x^{**}|_{(X^*)_{A^*}^{\beta}}, \quad x^{**} \in [(X^*)_{A^*}^{\alpha}]^*,$$

is injective and has a dense image in $[(X^*)_{A^*}^{\beta}]^*$. This image $[(X^*)_{A^*}^{\beta}]^*_{(X^*)_{A^*}^{\alpha}}$ consists of bounded linear functionals from $[(X^*)_{A^*}^{\beta}]^*$, which are also continuous in $(X^*)_{A^*}^{\beta}$ equipped with the norm $\|\cdot\|_{(X^*)_{A^*}^{\alpha}}$.

Denoting by $J\colon X \to X^{**}$ the canonical injection, which by the reflexivity of X is an isometry, we define the injective map $j_{\alpha} := (i_{*,\alpha,0})^* \circ J \in \mathcal{L}(X, [(X^*)_{A^*}^{\alpha}]^*)$. We have

$$\langle j_{\alpha}x, x^* \rangle_{[(X^*)_{A^*}^{\alpha}]^*,(X^*)_{A^*}^{\alpha}} = \langle x^*, x \rangle_{X^*,X}, \quad x^* \in (X^*)_{A^*}^{\alpha}, \quad x \in X.$$

Note that $j_0 = J$ and $j_{\alpha}(X) = [(X^*)_{A^*}^{\alpha}]^*_{X^*}$. For $\beta \in [0, \alpha]$, the latter subspace is injected in any $[(X^*)_{A^*}^{\alpha}]^*_{(X^*)_{A^*}^{\beta}}$, which is dense in $[(X^*)_{A^*}^{\alpha}]^*$ and for $\beta = \alpha$ coincides with $[(X^*)_{A^*}^{\alpha}]^*$.

Proposition 2.24 *For* $\alpha \geq 0$ *the map* j_{α} *is an isometric embedding between* X *normed by* $\left\|A^{-\alpha}\cdot\right\|_X$ *and* $[(X^*)_{A^*}^{\alpha}]^*$ *endowed with the standard norm*

$$\left\|x^{**}\right\|_{[(X^*)_{A^*}^{\alpha}]^*} = \sup\left\{\left|\langle x^{**}, x^* \rangle_{[(X^*)_{A^*}^{\alpha}]^*,(X^*)_{A^*}^{\alpha}}\right| : x^* \in (X^*)_{A^*}^{\alpha}, \left\|(A^*)^{\alpha}x^*\right\|_{X^*} \leq 1\right\}$$

and

$$\langle j_\alpha x, x^*\rangle_{[(X^*)^\alpha_{A^*}]^*,(X^*)^\alpha_{A^*}} = \langle x^*, x\rangle_\alpha =: \langle (A^*)^\alpha x^*, A^{-\alpha}x\rangle_{X^*,X}, \quad x^* \in (X^*)^\alpha_{A^*}, \quad x \in X.$$

Moreover, the image $j_\alpha(X)$ *is dense in* $[(X^*)^\alpha_{A^*}]^*$. *Hence* $[(X^*)^\alpha_{A^*}]^*$ *is a completion of* $(X, \left\|A^{-\alpha}\cdot\right\|_X)$. *In particular, we have*

$$\|j_\alpha x\|_{[(X^*)^\alpha_{A^*}]^*} = \left\|A^{-\alpha}x\right\|_X, \quad x \in X.$$

Taking $\alpha = 1$ in the above proposition, we see that the isometric embedding $j = j_1$ allows to construct one-sided fractional power scale generated by A_{-1}.

2.3 Extrapolated Fractional Power Scale in a Reflexive Banach Space

The equations of type (2.1) can be considered in a Banach space for which the operator $\mathcal{A}$ is realized as a positive sectorial operator $A \colon X \supseteq \mathrm{dom}(A) \to X$ and hence in any space X^α_A from the one-sided fractional power scale generated by A, that is, a *subspace* of the base space X. However, using the extrapolated fractional power scale one can realize the operator even in a bigger space using its adjoint operator, as it has already been observed by many authors, see e.g. [30]. A comprehensive description of the extrapolated fractional power scale can be found e.g. in [38, p. 14], where the presentation follows the abstract approach of [1]. We outline it briefly below.

We consider a positive operator $A \colon X \supseteq \mathrm{dom}(A) \to X$ in a reflexive Banach space X. Using the isometric embedding $j = j_1 = (i_{*,1,0})^* \circ J \in \mathcal{L}(X, [(X^*)^1_{A^*}]^*)$, we construct the extrapolated operator $A_{-1} \colon X^{-1} \supseteq j(X) \to X^{-1}$, where we denoted

$$X^{-1} = [(X^*)^1_{A^*}]^*, \quad \|x\|_{-1} = \|x\|_{[(X^*)^1_{A^*}]^*}, \quad x \in X^{-1}.$$

Note that $j(X) = [(X^*)^1_{A^*}]^*_{X^*}$ is a subspace of X^{-1} consisting of functionals from X^{-1} continuous with respect to the $\|\cdot\|_{X^*}$ −norm. Moreover, $\|jx\|_{-1} = \left\|A^{-1}x\right\|_X$ for $x \in X$. Thus there exists a one-sided fractional power scale $((X^{-1})^\alpha_{A_{-1}}, \|\cdot\|_{\alpha,A_{-1}}, (A_{-1})_\alpha)_{\alpha\geq 0}$ generated by A_{-1}.

We define a family of triples

$$(X^\alpha, \|\cdot\|_\alpha, A_{\alpha,-1})_{\alpha\geq -1} = ((X^{-1})^{\alpha+1}_{A_{-1}}, \|\cdot\|_{\alpha+1,A_{-1}}, (A_{-1})_{\alpha+1})_{\alpha\geq -1},$$

which we call *extrapolated fractional power scale generated by* A. The spaces X^α, $\alpha \geq -1$, are reflexive Banach spaces equipped with the norm

$$\|x\|_\alpha = \left\|(A_{-1})^{\alpha+1}x\right\|_{-1}, \quad x \in X^\alpha. \tag{2.11}$$

All spaces X^α, $\alpha \geq -1$, are subspaces of X^{-1} and X^β is densely and continuously injected into X^α provided that $-1 \leq \alpha \leq \beta$. Note also that the moment inequality

$$\|x\|_\beta \leq c\,\|x\|_\gamma^{\frac{\alpha-\beta}{\alpha-\gamma}}\,\|x\|_\alpha^{\frac{\beta-\gamma}{\alpha-\gamma}}, \quad x \in X^\alpha, \quad -1 \leq \gamma < \beta < \alpha, \tag{2.12}$$

holds true. The operator $A_{\alpha,-1}\colon X^\alpha \supseteq X^{\alpha+1} \to X^\alpha$ for $\alpha \geq -1$ is such that

$$A_{\alpha,-1}x = A_{-1}x, \quad x \in X^{\alpha+1}.$$

Denoting the injection $i_{\beta,\alpha}\colon X^\beta \overset{d}{\hookrightarrow} X^\alpha$, we see that

$$A_{\alpha,-1} \circ i_{\beta+1,\alpha+1} = i_{\beta,\alpha} \circ A_{\beta,-1} \quad \text{for} \quad -1 \leq \alpha < \beta.$$

We also have $\rho(A_{\alpha,-1}) = \rho(A_{-1}) = \rho(A)$ and $R(\lambda\colon A_{\alpha,-1}) = R(\lambda\colon A_{-1})|_{X^\alpha}$ for $\lambda \in \rho(A_{\alpha,-1})$. In particular, $A_{\alpha,-1}$ is a positive operator and

$$(A_{\alpha,-1})^\gamma \in \mathcal{L}iso(X^{\alpha+\gamma}, X^\alpha) \quad \text{and} \quad (A_{\alpha,-1})^\gamma = (A_{-1})^\gamma|_{X^{\alpha+\gamma}}, \quad \gamma \geq 0.$$

If A is a positive sectorial operator, then, given $\alpha \geq -1$, $A_{\alpha,-1}$ in X^α is also positive sectorial. Moreover, $e^{-A_{\alpha,-1}t} = e^{-A_{-1}t}|_{X^\alpha}$ and there exists $a > 0$ such that for $\gamma \geq 0$ we have with some $C_\gamma > 0$

$$\left\| e^{-A_{\alpha,-1}t} \right\|_{\mathcal{L}(X^\alpha, X^{\alpha+\gamma})} \leq \frac{C_\gamma e^{-at}}{t^\gamma}, \quad t > 0.$$

Using Lemma 2.22, for $\alpha \geq 0$ the spaces X^α are characterized in terms of the previously defined spaces as

$$X^\alpha = j(X_A^\alpha) \quad \text{and} \quad \|jx\|_\alpha = \|x\|_{\alpha,A} = \left\|A^\alpha x\right\|_X \quad \text{for } x \in X_A^\alpha,$$

whereas $A_{\alpha,-1}\colon j(X_A^\alpha) \supseteq j(X_A^{\alpha+1}) \to j(X_A^\alpha)$ and

$$A_{\alpha,-1} jx = A_{-1} jx = jAx, \quad x \in X_A^{\alpha+1} = \mathrm{dom}(A^{\alpha+1}).$$

In particular, we have $X^0 = j(X) = [(X^*)^1_{A^*}]^*_{X^*}$ and $A_{0,-1} jx = jAx$, $x \in \mathrm{dom}(A)$.

The spaces X^α for $\alpha \in (-1, 0)$ have the following characterization.

Lemma 2.25 *$X^{-\beta}$ for $\beta \in (0,1)$ is a completion of $(X, \left\|A^{-\beta}\cdot\right\|_X)$ via $j\colon X \to X^{-\beta}$.*

Summarizing, for $-1 \leq \alpha < \beta < 0 < \gamma < \delta$ we have

$$\begin{aligned} X^\delta &= j(X_A^\delta) \overset{d}{\hookrightarrow} X^\gamma = j(X_A^\gamma) \overset{d}{\hookrightarrow} X^0 = j(X) \\ &= (i_{*,1,0})^*(X^{**}) \overset{d}{\hookrightarrow} X^\beta \cong [(X^*)^{-\beta}_{A^*}]^* \overset{d}{\hookrightarrow} X^\alpha \overset{d}{\hookrightarrow} X^{-1} = [(X^*)^1_{A^*}]^*, \end{aligned}$$

where each space has a specific norm $\|\cdot\|_\sigma$, $\sigma \in \{0, \alpha, \beta, \gamma, \delta\}$, which makes it a reflexive Banach space, where

$$\|jx\|_\sigma = \left\|A^\sigma x\right\|_X, \quad x \in X^\sigma_A, \quad \sigma \in \{0, \gamma, \delta\}, \quad \|x\|_\sigma = \left\|A^\sigma x\right\|_X, \quad x \in X, \quad \sigma \in \{\alpha, \beta\}. \tag{2.13}$$

Moreover, the fractional power of order $\gamma \geq 0$ of $A_{\alpha,-1}$, being a restriction of $(A_{-1})^\gamma$ to $X^{\alpha+\gamma}$, takes this space isometrically onto X^α, that is,

$$(A_{\alpha,-1})^\gamma = (A_{-1})^\gamma \in \mathcal{L}iso(X^{\alpha+\gamma}, X^\alpha), \quad \alpha \geq -1, \quad \gamma \geq 0. \tag{2.14}$$

Concerning (2.10) and (2.14), we also refer the reader to [29, (3.9)] and [43, Sect. 1.15.2].

Iterating the construction of extrapolated operators, for a chosen $N \in \mathbb{N}$, a fractional power scale $(X^\alpha, \|\cdot\|_\alpha, A_{\alpha,-N})_{\alpha \geq -N}$ can be defined on the interval $[-N, \infty)$, for details see [1, Sect. V.1.3], but one has to be careful choosing the appropriate isometric embeddings (see [1, p. 263]). Since in our applications we do not go beyond the space $[(X^*)^1_{A^*}]^*$, we restrict our presentation to the case $N = 1$ as described above. Moreover, in the applications we will identify X with $j(X)$ and use Banach spaces isometric to X^α. Finally, to simplify the notation, considering an extrapolated fractional power scale, instead of $A_{\alpha,-1}$ we just write A_α, which should not be confused with an operator from the one-sided fractional power scale.

2.4 Solutions of Sectorial Equations

Interpreting the linear differential operator $\mathcal{A}$ from (2.1) in a suitable (reflexive) Banach space X as a positive sectorial operator A in X, the construction of fractional power scales described above gives us a natural family of Banach spaces $(X^\alpha, \|\cdot\|_\alpha)_{\alpha \in J}$, J being the interval $[0, \infty)$ in the case of one-sided or $[-1, \infty)$ in the case of extrapolated fractional power scale, in which the connected with that operator *linear abstract Cauchy problem*

$$u_t + A_\beta u = 0, \quad u(0) = u_0,$$

will be considered in the base space X^β, $\beta \in J$. Such a choice of spaces is specific for the operator A. Very often it will coincide with a family of closed subspaces of the Sobolev spaces, as seen in the examples described further in the text. It is also well-known that even in the simplest case, when A is a realization of a uniformly elliptic operator, the spaces obtained in this way may not in general coincide with the whole standard Sobolev spaces, being their proper closed linear subspaces.

Having established the fractional power scale, we verify that the nonlinearity $\mathcal{F}$ in (2.1) gives rise to its Nemitskii operator $F \colon X^\alpha \to X^\beta$ with some $\alpha, \beta \in J$, $0 \leq \alpha - \beta < 1$, which is Lipschitz continuous on bounded subsets of X^α. Thus we consider the *semilinear abstract Cauchy problem*

$$u_t + A_\beta u = F(u), \quad t > 0, \quad u(0) = u_0, \tag{2.15}$$

with the first equation satisfied in the base space X^β. Note that the operator A_β takes isometrically $X^{\beta+1}$ onto X^β, cp. (2.10) or (2.14), and we have $X^{\beta+1} \overset{d}{\hookrightarrow} X^\alpha \overset{d}{\hookrightarrow} X^\beta$. By the theory of D. Henry (see [6, 24]) for each $u_0 \in X^\alpha$ there exists a unique (X^α, X^β) solution of (2.15), that is, a function in the class

$$u \in C([0,\tau); X^\alpha) \cap C((0,\tau); X^{\beta+1}), \quad u_t \in C((0,\tau); X^{(\beta+1)-}),$$

defined on the maximal interval of existence and satisfying the first equation in (2.15) in X^β and the initial condition in (2.15) in X^α. In particular, the Duhamel formula

$$u(t) = e^{-A_\beta t}u_0 + \int_0^t e^{-A_\beta(t-s)}F(u(s))ds, \quad t \in [0,\tau),$$

holds, where $e^{-A_\beta t}$ is the linear analytic semigroup generated by $-A_\beta$, which coincides with $T(t) = e^{-At}$ or with $T(t) = e^{-A_{-1}t}$ on X^β, depending on the scale we consider, see also [29, (3.10)–(3.11)]. In particular, we have

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))ds, \quad t \in [0,\tau), \tag{2.16}$$

and for some $a > 0$ the estimates

$$\|T(t)\|_{\mathcal{L}(X^\gamma,X^\delta)} \leq \frac{C_{\gamma,\delta}e^{-at}}{t^{\delta-\gamma}}, \quad t > 0, \quad \delta \geq \gamma, \quad \delta,\gamma \in J, \tag{2.17}$$

hold with some $C_{\gamma,\delta} > 0$, see (2.7).

Note that the solutions to the problem (2.15) and, more generally, to (2.16) can be considered in a generalized sense for less regular initial data u_0 than from X^α if the linear semigroup satisfies (2.17) in some scale of Banach spaces. For details of this approach we refer the reader to [40] for the linear equations and to [8, 38] for the nonlinear problems with applications. In the latter paper there was introduced and thoroughly discussed the notion of a *γ-solution* to (2.16), (2.17), hence a generalization of the notion of a *mild solution* to the semilinear problem (2.15), requiring that the nonlinearity fulfills

$$F\colon X^\alpha \to X^\beta \quad \text{for some } \alpha,\beta \in J, \quad 0 \leq \alpha - \beta < 1,$$

and there exist $\rho \geq 1$ and $L > 0$ such that

$$\|F(u) - F(v)\|_\beta \leq L(1 + \|u\|_\alpha^{\rho-1} + \|v\|_\alpha^{\rho-1})\|u - v\|_\alpha, \quad u, v \in X^\alpha.$$

Under this particular form of local Lipschitz continuity of F more accurate estimates for solutions of the corresponding integral equation (2.16) are possible without assuming analyticity nor continuity of the linear semigroup $T(t)$ at $t = 0$, that is, for $u_0 \in X^\gamma$, $\gamma \in J$, $T(t)u_0 \to u_0$ in X^γ as $t \to 0^+$.

The following definition of the γ-solution is used in that reference.

Definition 2.26 If $u_0 \in X^\gamma$ then a function $u \in L^\infty_{loc}((0,\tau]; X^\alpha)$ such that $t^{\alpha-\gamma}\|u\|_{X^\alpha} \le M$, $t \in (0,\tau]$ with certain $M > 0$, satisfying $u(0) = u_0$ and

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))ds, \quad 0 < t \le \tau,$$

is called a γ-*solution of* (2.16) *in* $[0,\tau]$.

Using the smoothing action of the integral equation, the authors are able to set initial data in a larger space X^γ on the scale than X^α. More precisely, they may have $u_0 \in X^\gamma$ with the parameter γ satisfying

$$\gamma \in \left(\alpha - \frac{1}{\rho}, \alpha\right] \text{ if } 0 \le \alpha - \beta \le \frac{1}{\rho}, \quad \text{and} \quad \gamma \in \left[\frac{\alpha\rho - \beta - 1}{\rho - 1}, \alpha\right] \text{ if } \frac{1}{\rho} < \alpha - \beta < 1.$$

2.5 Domains of Fractional Powers of Negative Dirichlet Laplacian

We next recall the description of the domains of fractional powers $(-\Delta_{L^p})^\alpha$, $\alpha \ge 0$, of the negative Dirichlet Laplacian realized in $X = L^p(\Omega)$, $1 < p < \infty$, in a bounded $C^{2\lceil\alpha\rceil}$ domain Ω, where $\lceil\alpha\rceil$ denotes the least integer greater or equal to α. In the Hilbert case $p = 2$ such description was discussed in detail in [29, p. 303] and in [45, Sect. 16.6], see also [6, Sect. 1.3.3]. It follows that $D((-\Delta_{L^2})^\alpha)$ for $\alpha \ge 0$ is a subspace of the Sobolev–Lebesgue space $H^{2\alpha}(\Omega)$ (cp. [45, Sect. 1.11]) and we have, in particular,

$$D((-\Delta_{L^2})^\alpha) = \begin{cases} H^{2\alpha}(\Omega) & \text{if } 0 \le \alpha < \frac{1}{4}, \\ H^{2\alpha}_{\{Id\}}(\Omega) & \text{if } \frac{1}{4} < \alpha < \frac{5}{4}, \alpha \ne \frac{3}{4}, \\ H^{2\alpha}_{\{Id,\Delta\}}(\Omega) & \text{if } \frac{5}{4} < \alpha < \frac{9}{4}, \alpha \ne \frac{7}{4}, \end{cases} \tag{2.18}$$

where $H^{2\alpha}_{\{Id\}}(\Omega)$, later in the text also denoted by $H^{2\alpha}_0(\Omega)$, stands for the subspace of $H^{2\alpha}(\Omega)$ consisting of functions with zero value on $\partial\Omega$ (in the sense of trace) and $H^{2\alpha}_{\{Id,\Delta\}}(\Omega)$ stands for the subspace of $H^{2\alpha}(\Omega)$ consisting of elements v which satisfy $v = \Delta v = 0$ on $\partial\Omega$ (in the sense of trace).

In case of sectorial operators in the Lebesgue space $L^p(\Omega)$, $1 < p < \infty$, such description can be found in [24, Theorem 1.6.1] when $\partial\Omega \in C^{2\lceil\alpha\rceil}$, or in more detail in [45, pp. 81, 560]. In particular, for uniformly strongly elliptic operators in the $L^p(\Omega)$ setting under the assumptions of [45, Theorem 16.14] (which are satisfied for the negative Laplacian in part regarding the boundedness of imaginary powers thanks to [37, p. 166]) we have $D((-\Delta_{L^p})^\alpha) \subseteq W^{2\alpha,p}(\Omega)$, $\alpha \ge 0$, and, in particular,

$$D((-\Delta_{L^p})^\alpha) = \begin{cases} W^{2\alpha,p}(\Omega) & \text{if } 0 \le \alpha < \frac{1}{2p}, \\ W^{2\alpha,p}_{\{Id\}}(\Omega) & \text{if } \frac{1}{2p} < \alpha < 1 + \frac{1}{2p}, \alpha \ne \frac{1}{2} + \frac{1}{2p}, \\ W^{2\alpha,p}_{\{Id,\Delta\}}(\Omega) & \text{if } 1 + \frac{1}{2p} < \alpha < 2 + \frac{1}{2p}, \alpha \ne \frac{3}{2} + \frac{1}{2p}. \end{cases} \tag{2.19}$$

The space $W^{2\alpha,p}_{\{Id\}}(\Omega)$, later in the text also denoted by $W^{2\alpha,p}_0(\Omega)$, is a subspace of the Sobolev–Lebesgue space $W^{2\alpha,p}(\Omega)$ and consists of functions with zero value on $\partial\Omega$ (in the sense of trace). Even more complete description of the domains of fractional powers of negative Dirichlet Laplacian (and more general elliptic operators) was known much earlier, see [43, Sect. 5.5.1]. However, those results were formulated under the assumption of C^∞ regularity of the domain Ω and the coefficients of the differential operator involved. A lot of effort was needed to weaken those too high regularity assumptions (see e.g. [13, 17, 34, 37]). An extended discussion of the domains of fractional powers of sectorial operators corresponding to second order uniformly strongly elliptic operators in bounded smooth domains $\Omega \subset \mathbb{R}^N$ with Dirichlet or Neumann boundary conditions was also reported in [29] in case of the Hilbert scales.

Remark 2.27 Considering higher fractional powers of the negative Dirichlet Laplacian, we deal with the so called *Navier boundary conditions*. More precisely, the functions u from the domains of higher powers $(-\Delta)^k$, $k = 1, 2, \ldots$, considered in a bounded domain Ω with $\partial\Omega \in C^{2k}$, satisfy the Navier boundary conditions, see [35], given by

$$u\,|_{\partial\Omega} = \Delta u\,|_{\partial\Omega} = \cdots = \Delta^{k-1} u\,|_{\partial\Omega} = 0.$$

The situation presented in [43, Theorem 5.4.4/2] for a regular elliptic operator $\mathcal{A}$ of $2m$th order and m boundary conditions $B_1, \ldots, B_m$, and realized as a sectorial positive operator A in the space $L^p(\Omega)$, $1 < p < \infty$, is, however, slightly different. Besides the (convenient) C^∞ regularity assumption on $\partial\Omega$, the result formulated in [43] states that the operator

$$A^{(s)}u = \mathcal{A}u, \quad s = 0, 1, \ldots, \quad \text{with } D(A^{(s)}) = W^{2m+s}_{p,\{B_j\}}(\Omega), \tag{2.20}$$

acts as a bijective bounded operator from $W^{2m+s}_{p,\{B_j\}}(\Omega)$ onto $W^s_p(\Omega)$ (with notation of [43]). However, the functions from the fractional power space $D(A^\alpha)$ for $\alpha \in \mathbb{N}$ require as many as αm boundary conditions (see [6, Remark 1.3.7]). Additionally, the operator A takes isometrically $D(A^{\alpha+1})$ onto $D(A^\alpha)$ (see (2.10)), making it different from the action of (2.20).

2.6 Global Extendibility of Small Data Solutions for Super-Linear Nonlinearity

Let the realization of the linear differential operator $\mathcal{A}$ in (2.1) be a selfadjoint and positive definite operator A in a Hilbert space X, so that the (one-sided or extrapolated) fractional power scale X^α, $\alpha \in J$, consists of Hilbert spaces normed by $\|\cdot\|_\alpha$ given in (2.8) or (2.11), cp. also (2.13). For more details of the Hilbert fractional power scale we refer the reader e.g. to [29], [1, Theorems III.4.6.7, V.1.5.15], [42, p. 133] and [20, Sect. 5.2.2].

Assume that the nonlinearity $\mathcal{F}$ in (2.1) defines the Nemitskii operator $F\colon X^\alpha \to X^\beta$ with some $\alpha, \beta \in J$, $0 \le \alpha - \beta < 1$, which is Lipschitz continuous on bounded subsets of X^α. We consider the semilinear problem (2.15) with the first equation satisfied in the base space X^β. For each $u_0 \in X^\alpha$ there exists a unique (X^α, X^β)

solution of (2.15),

$$u \in C([0,\tau); X^{\alpha}) \cap C((0,\tau); X^{\beta+1}), \quad u_t \in C((0,\tau); X^{(\beta+1)-}),$$

defined on the maximal interval of existence and satisfying the first equation in (2.15) in X^{β} and the initial condition in (2.15) in X^{α}. In particular, it satisfies the Duhamel formula (2.16) with $T(t) = e^{-At}$ or $T(t) = e^{-A_{-1}t}$ being an analytic semigroup for which the estimates (2.17) hold.

We further assume that the following super-linear *growth condition* holds

$$\|F(u)\|_{\beta} \le a\|u\|_{\beta+1}^{\nu}\|u\|_{\beta+\frac{1}{2}}^{\mu} + b, \quad u \in X^{\beta+1}, \tag{2.21}$$

where $a, b \ge 0$, $\nu \in [0,1)$ and $\mu + \nu > 1$.

We next formulate a result on the global in time a priori estimate in $X^{\beta+\frac{1}{2}}$ valid for small data solutions (see [20, Sect. 6.2] for a prototype of this estimate for the 3D Navier–Stokes equation).

Lemma 2.28 *Let condition (2.21) be satisfied with $b \ge 0$ such that (2.24) holds and let u be an (X^{α}, X^{β}) solution of the problem (2.15) with small initial data $u_0 \in X^{\beta+\frac{1}{2}}$, as specified in (2.25). Then such solutions are bounded a priori in $X^{\beta+\frac{1}{2}}$ norm.*

Proof We carry out the proof for the extrapolated fractional power scale. Under the conditions of the theorem, the extrapolated operator A_{-1} is selfadjoint and positive definite in the Hilbert space X^{-1}, see [1, Theorem V.1.5.15]. We apply the operator $(A_{-1})^{\beta+1}\colon X^{\beta} \to X^{-1}$ to the first equation in (2.15) and take the scalar product $\langle\cdot,\cdot\rangle$ in X^{-1} with $(A_{-1})^{\beta+2}u$, getting

$$\langle (A_{-1})^{\beta+1}u_t, (A_{-1})^{\beta+2}u\rangle + \langle (A_{-1})^{\beta+2}u, (A_{-1})^{\beta+2}u\rangle = \langle (A_{-1})^{\beta+1}F(u), (A_{-1})^{\beta+2}u\rangle.$$

Using the fact that A_{-1} is selfadjoint and the growth condition (2.21), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{\beta+\frac{1}{2}}^2 + \|u\|_{\beta+1}^2 \le a\|u\|_{\beta+1}^{1+\nu}\|u\|_{\beta+\frac{1}{2}}^{\mu} + b\|u\|_{\beta+1}, \quad t > 0. \tag{2.22}$$

Applying to the penultimate term the Young inequality

$$a\|u\|_{\beta+1}^{1+\nu}\|u\|_{\beta+\frac{1}{2}}^{\mu} \le \varepsilon\|u\|_{\beta+1}^2 + C_{\varepsilon}\|u\|_{\beta+\frac{1}{2}}^{\frac{2\mu}{1-\nu}},$$

with $\varepsilon = \frac{1}{4}$, and to the last term of (2.22) the Cauchy inequality, it extends to

$$\frac{d}{dt}\|u\|_{\beta+\frac{1}{2}}^2 \le -\|u\|_{\beta+1}^2 + C\|u\|_{\beta+\frac{1}{2}}^{\frac{2\mu}{1-\nu}} + 2b^2$$

with a certain constant $C = C(a,\mu,\nu) > 0$.

Next, using the generalized Poincaré inequality (see e.g. [20, p. 115])

$$\lambda_1 \|(A_{-1})^{\beta+\frac{3}{2}}\phi\|_{-1}^2 \leq \|(A_{-1})^{\beta+2}\phi\|_{-1}^2 \quad \text{for } \phi \in X^{\beta+1},$$

where $\lambda_1 > 0$ stands for the first positive eigenvalue of the operator A_{-1}, we find that

$$\frac{d}{dt}\|u\|_{\beta+\frac{1}{2}}^2 \leq -\lambda_1 \|u\|_{\beta+\frac{1}{2}}^2 + C\|u\|_{\beta+\frac{1}{2}}^{\frac{2\mu}{1-\nu}} + 2b^2, \quad t > 0. \tag{2.23}$$

Analyzing the real function in the right-hand side of the above differential inequality,

$$g_b(y) = -\lambda_1 y + Cy^{\frac{\mu}{1-\nu}} + 2b^2,$$

where y corresponds to $\|u\|_{\beta+\frac{1}{2}}^2$, we see that $g_b(0) = 2b^2$ with $g_b'(0) = -\lambda_1 < 0$, and the minimal value of g_b is attained at the point $y_{min} = \left(\lambda_1 \frac{1-\nu}{C\mu}\right)^{\frac{1-\nu}{\mu+\nu-1}}$ and

$$g_b(y_{min}) = \tfrac{C}{1-\nu}(y_{min})^{\frac{\mu}{1-\nu}}(1-\nu-\mu) + 2b^2.$$

Choosing

$$b^2 < \tfrac{C}{2(1-\nu)}(y_{min})^{\frac{\mu}{1-\nu}}(\mu+\nu-1), \tag{2.24}$$

we get $g_b(y_{min}) < 0$, with the graph of g_b having two nonnegative zeros; $0 \leq y_1 < y_2$ (note that $y_1 = 0$ when $b = 0$). Consequently, for such a small b^2, it follows from (2.23) that if

$$\|u_0\|_{\beta+\frac{1}{2}}^2 \leq y_2, \tag{2.25}$$

then $\|u(t)\|_{\beta+\frac{1}{2}}^2 \leq y_2$ for $t \geq 0$, hence the norm $\|u(t)\|_{\beta+\frac{1}{2}}$ is bounded uniformly in time. □

In particular, this lemma guarantees the global in time extendibility of the local $\left(X^{\beta+\frac{1}{2}}, X^{\beta}\right)$ solutions of (2.15) corresponding to small initial data u_0 satisfying (2.25) whenever the growth condition (2.21) is satisfied with $b \geq 0$ sufficiently small [see (2.24)].

We also recall [15, Theorem 4.2] suitably modified for our purposes.

Theorem 2.29 *Assume that the following a priori estimate for the* (X^α, X^β) *solution* $u(t)$ *of* (2.15) *satisfying* $u(0) = u_0 \in X^\alpha$ *holds in a normed space* Y *such that* $Y \hookrightarrow X^\alpha$, *that is, there exists a function* $c\colon [0, T) \to [0, \infty)$, $0 < T \leq \infty$, *bounded on compact intervals and such that*

$$\|u(t)\|_Y \leq c(t), \quad t \in (0, \min\{\tau_{u_0}, T\}), \tag{2.26}$$

where τ_{u_0} *denotes the life time of the solution. Furthermore, assume that the following* subordination condition *holds for the nonlinearity, that is, there exist a nondecreasing* *function* $g\colon [0,\infty) \to [0,\infty)$ *and a constant* $\theta \in [0,1)$ *such that*

$$\|F(u(t))\|_\beta \le g(\|u(t)\|_Y)\left(1+\|u(t)\|_\alpha^\theta\right), \quad t \in (0,\tau_{u_0}). \tag{2.27}$$

Then we have $\tau_{u_0} \ge T$.

If $T=\infty$ in the a priori estimate (2.26), then the solution of (2.15) from Theorem 2.29 exists globally in time. Moreover, if $T=\infty$ in (2.26) and the function $c(t)$ is bounded on $[0,\infty)$ by some constant, then this solution of (2.15) exists globally in time and is bounded.

As will be seen from the analysis of the examples in Sects. 3 and 4, for many nonlinearities in the physical models (in particular, in the 2D quasi-geostrophic equation) there is a limitation from above of the level on the fractional power scale at which the nonlinearity will be realized. This is connected with the required compatibility conditions to be satisfied, see [28] for considerations concerning this issue. Consequently, for such equations we are not able to construct 'too regular' solutions in the formalism of this paper. But there is also another limitation, from below, for the admissible 'too weak' regularity of solutions, connected with fact that the nonlinearity will not behave well on such less regular solutions. More precisely, the Nemitskii superposition operator (see e.g. [2] or [9, Sect. 2.5]) associated with the nonlinearity will not be well-defined.

In general, it is not easy to relate the nonlinearity with the fractional power scale generated by the linear operator in the equation. However, for example, the semilinear parabolic Dirichlet problem considered in a bounded regular domain $\Omega \subset \mathbb{R}^N$

$$\begin{aligned} &u_t = \Delta u + f(u,\nabla u), \quad x \in \Omega \subset \mathbb{R}^N, \quad t>0,\\ &u=0 \text{ on } \partial\Omega, \quad u(0,x)=u_0(x), \quad x\in\Omega, \end{aligned} \tag{2.28}$$

fits well into the scale generated by $-\Delta$ because the gradient ∇u is of the same order as $(-\Delta)^{\frac{1}{2}}u$ and it is not hard to formulate the admissible growth restriction on the nonlinearity leading to global in time solutions of (2.28). In particular, we can assume that

$$|f(u,\nabla u)| \le g(u) + b|\nabla u|^{2-\varepsilon} \quad \text{with small } \varepsilon>0,$$

where the admissible growth exponent $2-\varepsilon$ follows from the moment inequality

$$\|\phi\|_{H_0^1(\Omega)} \le c\|(-\Delta)^{\frac{1}{2}}\phi\|_{L^2(\Omega)} \le c\|\phi\|_{H^{2-\varepsilon}(\Omega)}^{\frac{1}{2-\varepsilon}}\|\phi\|_{L^2(\Omega)}^{\frac{1-\varepsilon}{2-\varepsilon}}.$$

In low space dimensions $N \le 3$, having the embedding $H^{2-\varepsilon}(\Omega) \hookrightarrow L^\infty(\Omega)$, it is sufficient to take $g\colon \mathbb{R}\to\mathbb{R}$ locally Lipschitz continuous. For $N \ge 4$, we need to limit eventually the growth of g to stay inside the scale generated by $-\Delta$ on $L^2(\Omega)$. Alternatively, we can switch into the scale generated by $-\Delta$ realized in $L^p(\Omega)$, $p>N$, to avoid implementing additional growth restrictions on g.

3 Dirichlet Problem for 2D Quasi-geostrophic Equation

This and the following section are devoted to a discussion of two examples of semi-linear equations, which can be solved at various levels of the fractional power scale generated by the positive sectorial operator in their main part. Here we will concentrate on the 2D quasi-geostrophic equation with Dirichlet boundary conditions, whereas in the next section we discuss the fractional chemotaxis equation, with recent results reported in [3, 19, 20, 44]. The approach to these problems is unified and can be used to treat other, even more complicated, problems as well. For instance, in separate papers [7, 18] the celebrated 3D Navier–Stokes equation on the extrapolated fractional power scale was discussed in detail. Let us also mention that studying admissible levels of the extrapolated fractional power scale at which the examples can be settled, we use a similar reasoning to the one originated in [22, Lemma 2.2] in case of the Navier–Stokes equation.

In this section we consider the Dirichlet problem in a bounded C^2 domain for the 2D quasi-geostrophic equation (see e.g. [11, 19, 20]). Frequently studied earlier Cauchy problem in $\mathbb{R}^2$ for that equation, with a positive parameter ω, has the form

$$\begin{aligned} &\theta_t + u\cdot\nabla\theta + \kappa(-\Delta)^{\omega}\theta = f, \quad x\in\mathbb{R}^2, \quad t>0,\\ &\theta(0,x) = \theta_0(x), \quad x\in\mathbb{R}^2, \end{aligned} \tag{3.1}$$

where the velocity field $u = (u_1, u_2)$ is expressed by the stream function ψ through the relation

$$u = \left(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\right), \quad \text{where } (-\Delta)^{\frac{1}{2}}\psi = -\theta. \tag{3.2}$$

Here θ represents the potential temperature, $\kappa > 0$ is a diffusivity coefficient, f is a free time-independent force, and $(-\Delta)^{\omega}$ denotes a nonlocal operator

$$(-\Delta)^{\omega}\phi = \mathcal{F}^{-1}(|\xi|^{2\omega}\mathcal{F}(\phi)(\xi)),$$

which is an equivalent way to define ω-power of the negative Laplacian in $\mathbb{R}^N$, see [27].

Following the studies of e.g. [10, 19], we replace in the above $-\Delta$ and $(-\Delta)^{\omega}$ by the negative Dirichlet Laplacian in $L^2(\Omega)$ and its fractional power, respectively, and consider the counterpart of (3.1), (3.2) in the form of the Dirichlet problem for the quasi-geostrophic equation in a bounded domain Ω in $\mathbb{R}^2$ with C^2 boundary. Observe that the Formula (3.2) can be alternatively expressed using the components of the *Riesz transform* $\mathcal{R}$ in the bounded domain Ω in the form

$$u = \mathcal{R}^{\perp}\theta = \left(-\mathcal{R}_2\theta, \mathcal{R}_1\theta\right). \tag{3.3}$$

Rescaling the time variable, renaming f and plugging (3.3) into the equation leads to the abstract Cauchy problem, which we consider in this section

$$\begin{aligned}&\theta_t + (-\Delta)^{\omega}\theta = F(\theta) := -\kappa^{-1}\mathcal{R}^{\perp}\theta \cdot \nabla\theta + f,\\ &\theta(0) = \theta_0,\end{aligned} \tag{3.4}$$

with a parameter $\omega \in \left(\frac{1}{2}, 1\right]$.

Before addressing the question of solvability of (3.4), we recall some basic properties of the Riesz transform. By an analogy to the case of the whole of $\mathbb{R}^N$, see e.g. [33, Sect. 12.3, (12.22)], the Riesz transform in a bounded regular domain $\Omega \subset \mathbb{R}^N$ is defined through the formula

$$\mathcal{R} = -\nabla(-\Delta)^{-\frac{1}{2}}, \quad \text{with components } \mathcal{R}_j = -\frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}, \quad j = 1, 2, \dots, N,$$

for elements from the domain of σ-power of the negative Dirichlet Laplacian $D((-\Delta)^{\sigma})$), see Sect. 2.5.

Proposition 3.1 *The Riesz transforms $\mathcal{R}_j$, $j = 1, 2, \dots, N$, in a bounded $C^{2\lceil\sigma\rceil}$ domain Ω, are bounded operators from $D((-\Delta)^{\sigma})$ into $H^{2\sigma}(\Omega)$ for any $\sigma > 0$.*

Proof Note that if $v \in D((-\Delta)^{\sigma})$, $\sigma > 0$, then $(-\Delta)^{-\frac{1}{2}}v \in D\left((-\Delta)^{\sigma+\frac{1}{2}}\right)$ by (2.9). Furthermore, since any partial derivative is a bounded linear operator from $H^{s+1}(\Omega)$ to $H^s(\Omega)$ for $s \geq 0$ (cf. [45, Theorem 1.43]), we have $-\frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}v \in H^{2\sigma}(\Omega)$ for $j = 1, 2, \dots, N$. Consequently, we obtain by Proposition 2.7

$$\begin{aligned}\|\mathcal{R}_j v\|_{H^{2\sigma}(\Omega)} &= \|\frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}v\|_{H^{2\sigma}(\Omega)} \leq c\|(-\Delta)^{-\frac{1}{2}}v\|_{H^{2\sigma+1}(\Omega)}\\ &\leq c\|(-\Delta)^{\sigma+\frac{1}{2}}(-\Delta)^{-\frac{1}{2}}v\|_{L^2(\Omega)} = c\|(-\Delta)^{\sigma}v\|_{L^2(\Omega)} = c\|v\|_{D((-\Delta)^{\sigma})},\end{aligned}$$

where the equivalence of norms in $D((-\Delta)^s)$ and $H^{2s}(\Omega)$ was used. □

We also note that the Riesz transform is bounded in $L^p(\Omega)$, $1 < p < \infty$; see [41, Theorem C] in case of a bounded C^1 domain Ω.

Proposition 3.2 *If Ω is a bounded C^2 domain in $\mathbb{R}^N$, then*

$$\left\|\mathcal{R}_j v\right\|_{L^p(\Omega)} \leq c\,\|v\|_{L^p(\Omega)}, \quad v \in L^p(\Omega), \quad 2 \leq p < \infty, \quad j = 1, 2, \dots, N.$$

Proposition 3.1 shows that the Riesz operators $\mathcal{R}_j$ switch from the scale $D((-\Delta)^{\sigma})$ into the scale $H^{2\sigma}(\Omega)$, $\sigma \geq 0$. Indeed, they do not preserve the boundary condition (see also Remark 3.3). Thus, if we want to consider (3.4) in the scale $D((-\Delta)^{\sigma})$, there exists a maximal admissible regularity of the considered solutions of (3.4) and we are allowed to choose as the *base space* the space $D((-\Delta)^{\sigma})$ as far as it coincides with $H^{2\sigma}(\Omega)$, that is, whenever $\sigma < \frac{1}{4}$, see (2.18).

Remark 3.3 It was observed in [10] that for regular enough solutions, for example if $\theta \in H_0^1(\Omega)$, the normal component of the velocity vanishes at the boundary $\mathcal{R}^{\perp}\theta \cdot \nu \mid_{\partial\Omega} = 0$, because the stream function $\psi = -(-\Delta)^{-\frac{1}{2}}\theta$ vanishes at the boundary and its gradient is normal to the boundary. In case of a bounded C^2 domain Ω we cannot thus expect that the Riesz transforms $\mathcal{R}_j$ will vanish at the boundary of Ω.

We also remark that if $\Omega = \mathbb{R}^N$ the Riesz transforms $\mathcal{R}_j$ in the whole of $\mathbb{R}^N$ commute with the fractional powers of the negative Laplacian, for the proof see [33, (12.24)]. Consequently, we have the commutativity of the fractional powers with partial derivatives in $\mathbb{R}^N$. However, there is no similar property for the Riesz operator $\mathcal{R}$ in a bounded domain.

3.1 Solutions of (3.4) in the Base Space $L^2(\Omega)$.

We start with the local existence result for (3.4) with $\omega \in \left(\frac{1}{2}, 1\right]$. Using the approach of [6, 24], we rewrite (3.4) as

$$\theta_t + A\theta = F(\theta), \quad \theta(0) = \theta_0, \tag{3.5}$$

where $A = (-\Delta)^{\omega}$ and $F(\theta) = -\kappa^{-1}\mathcal{R}^{\perp}\theta \cdot \nabla\theta + f$ with $\omega \in \left(\frac{1}{2}, 1\right]$ and $\kappa > 0$. By the rule of exponentiation of powers of operators, see [25, Theorem 10.6] and [6, Proposition 1.3.7], we have $A^{\alpha} = (-\Delta)^{\omega\alpha}$ for $\alpha \geq 0$. Hence the one-sided fractional power scale X^{α}, $\alpha \geq 0$, generated by A, coincides with $D((-\Delta)^{\omega\alpha})$, $\alpha \geq 0$, the characterization of which is recalled in (2.18). In particular, the norm of X^{α},

$$\|\phi\|_{\alpha} = \left\|(-\Delta)^{\omega\alpha}\phi\right\|_{L^2(\Omega)}, \quad \phi \in X^{\alpha}, \tag{3.6}$$

and the norm in $H^{2\omega\alpha}(\Omega)$ are equivalent on $X^{\alpha} = D((-\Delta)^{\omega\alpha})$ for $\alpha \in [0, \frac{1}{\omega})$.

Considering $X^0 = L^2(\Omega)$ as the base space for (3.5), its solutions will vary in the phase space X^{α}, $\alpha \in \left(\frac{1}{2\omega}, 1\right)$. To justify the local in time solvability of (3.5), we need to check that for $f \in L^2(\Omega)$ the nonlinearity F is Lipschitz continuous on bounded sets as a map from X^{α} into $L^2(\Omega)$. Indeed, for θ_1, θ_2 in X^{α}, $\alpha \in \left(\frac{1}{2\omega}, 1\right)$, we have

$$\begin{aligned} \|F(\theta_1) - F(\theta_2)\|_{L^2(\Omega)} \leq \kappa^{-1} &\left\| \mathcal{R}_2(\theta_1 - \theta_2)\frac{\partial\theta_1}{\partial x_1} + \mathcal{R}_2\theta_2\frac{\partial(\theta_1 - \theta_2)}{\partial x_1} \right\|_{L^2(\Omega)} \\ &+ \kappa^{-1}\left\| \mathcal{R}_1(\theta_1 - \theta_2)\frac{\partial\theta_1}{\partial x_2} + \mathcal{R}_1\theta_2\frac{\partial(\theta_1 - \theta_2)}{\partial x_2} \right\|_{L^2(\Omega)}. \end{aligned} \tag{3.7}$$

We estimate the terms on the right-hand side of (3.7). Since $2\omega\alpha > 1$, using equivalent norms on the domains of fractional powers of $-\Delta$, we obtain by Proposition 3.1

$$\begin{aligned}\left\|\mathcal{R}_{3-j}(\theta_1-\theta_2)\frac{\partial\theta_1}{\partial x_j}\right\|_{L^2(\Omega)} &\le \|\mathcal{R}_{3-j}(\theta_1-\theta_2)\|_{L^\infty(\Omega)}\left\|\frac{\partial\theta_1}{\partial x_j}\right\|_{L^2(\Omega)}\\ &\le c\|\mathcal{R}_{3-j}(\theta_1-\theta_2)\|_{H^{2\omega\alpha}(\Omega)}\|\theta_1\|_{H^1(\Omega)}\\ &\le c\|\theta_1-\theta_2\|_\alpha\|\theta_1\|_\alpha, \quad j=1,2.\end{aligned}$$

By an analogous argument, we also get the estimate

$$\left\|\mathcal{R}_{3-j}\theta_2\frac{\partial(\theta_1-\theta_2)}{\partial x_j}\right\|_{L^2(\Omega)} \le c\|\theta_2\|_\alpha\|\theta_1-\theta_2\|_\alpha, \quad j=1,2.$$

Consequently, for each $\alpha \in \left(\frac{1}{2\omega}, 1\right)$, we obtain

$$\|F(\theta_1)-F(\theta_2)\|_{L^2(\Omega)} \le c\big(\|\theta_1\|_\alpha+\|\theta_2\|_\alpha\big)\|\theta_1-\theta_2\|_\alpha.$$

We have thus shown that $F\colon X^\alpha \to L^2(\Omega)$ is Lipschitz continuous on bounded sets, which proves the local solvability of (3.5) in the phase space $X^\alpha = D((-\Delta)^{\omega\alpha})$ endowed with the norm given in (3.6). Following [6, 24], we formulate a more precise result.

Theorem 3.4 *Let $\alpha \in \left(\frac{1}{2\omega}, 1\right)$ be fixed. Then, for $f \in L^2(\Omega)$ and any $\theta_0 \in X^\alpha \subseteq H_0^{2\omega\alpha}(\Omega)$, there exists a unique local in time (X^α, X^0) solution θ to the problem (3.5). Moreover, we have*

$$\theta \in C([0,\tau); D((-\Delta)^{\omega\alpha})) \cap C((0,\tau); D((-\Delta)^{\omega})), \quad \theta_t \in C((0,\tau); D((-\Delta)^{\gamma})),$$

with arbitrary $\gamma < \omega$, where $\tau > 0$ denotes the life time of that solution. Moreover, the Duhamel formula is satisfied

$$\theta(t) = e^{-At}\theta_0 + \int_0^t e^{-A(t-s)}F(\theta(s))ds, \quad t \in [0,\tau),$$

where e^{-At} denotes the linear semigroup generated by $-A = -(-\Delta)^\omega$ on $L^2(\Omega)$.

Our next goal is to extend the just constructed local (X^α, X^0) solutions with $\alpha \in \left(\frac{1}{2\omega}, 1\right)$ globally in time. For that purpose, we will use a version of a *subordination condition*. Indeed, we estimate

$$\|\mathcal{R}^\perp\theta\cdot\nabla\theta\|_{L^2(\Omega)} = \left\|-\mathcal{R}_2\theta\frac{\partial\theta}{\partial x_1}+\mathcal{R}_1\theta\frac{\partial\theta}{\partial x_2}\right\|_{L^2(\Omega)},$$

applying the Hölder inequality with $\varepsilon > 0$ and Proposition 3.2 in

$$\begin{aligned}\left\|\mathcal{R}_{3-j}\theta\frac{\partial\theta}{\partial x_j}\right\|_{L^2(\Omega)} &\leq \|\mathcal{R}_{3-j}\theta\|_{L^{\frac{4+2\epsilon}{\epsilon}}(\Omega)}\left\|\frac{\partial\theta}{\partial x_j}\right\|_{L^{2+\varepsilon}(\Omega)}\\ &\leq c\|\theta\|_{L^{\frac{4+2\varepsilon}{\varepsilon}}(\Omega)}\|\theta\|_{W^{1,2+\varepsilon}(\Omega)}, \quad j=1,2,\end{aligned}$$

which by the Sobolev embedding

$$H^{2\omega(\alpha-\delta)}(\Omega)\hookrightarrow W^{1,2+\varepsilon}(\Omega) \quad \text{with } \delta\in\left(0,\alpha-\frac{1}{2\omega}\right), \quad \varepsilon=\frac{2\omega\left(\alpha-\delta-\frac{1}{2\omega}\right)}{1-\omega\alpha+\omega\delta},$$

gives the following subordination condition:

$$\|\mathcal{R}^{\perp}\theta\cdot\nabla\theta\|_{L^2(\Omega)} \leq c\|\theta\|_{L^{\frac{4+2\varepsilon}{\varepsilon}}(\Omega)}\|\theta\|_{H^{2\omega(\alpha-\delta)}(\Omega)} \leq \tilde{c}\|\theta\|_{L^\infty(\Omega)}\|\theta\|_{\alpha-\delta}. \tag{3.8}$$

The uniform in ω a priori estimate

$$\|\theta(t)\|_{L^\infty(\Omega)} \leq \|\theta_0\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}, \tag{3.9}$$

is available for solutions of (3.5), see e.g. [19, (24)]. This a priori estimate in $L^\infty(\Omega)$ together with the subordination condition (3.8) is sufficient for the global in time extendibility of the local solutions constructed in Theorem 3.4.

Proposition 3.5 *Given $f \in L^\infty(\Omega)$, the solutions of (3.5) from Theorem 3.4 are global in time, i.e., $\tau = \infty$, and are bounded in the phase space $X^\alpha = D((-\Delta)^{\omega\alpha}) \subseteq H_0^{2\omega\alpha}(\Omega)$.*

Proof Applying the moment inequality (2.6) to the subordination condition (3.8), we get

$$\|F(\theta)\|_{L^2(\Omega)} \leq g(\|\theta\|_{L^\infty(\Omega)})\left(1+\|\theta\|_\alpha^{1-\frac{\delta}{\alpha}}\right), \quad t\in(0,\tau),$$

where g is a nondecreasing function. Combining this with the a priori estimate (3.9) in $L^\infty(\Omega)$, we apply Theorem 2.29 and conclude that $\tau = \infty$ and the solution θ is bounded in X^α. □

3.2 Solutions of (3.4) in a Smoother Base Space than $L^2(\Omega)$

As it was already mentioned, the Riesz operators do not preserve the boundary condition, so if we want to consider the 2D quasi-geostrophic equation in the base space X^β with $\beta \geq 0$ from the one-sided fractional power scale generated by $A = (-\Delta)^\omega$, our choice is limited to spaces for which $X^\beta = D((-\Delta)^{\omega\beta})$ coincides with $H^{2\omega\beta}(\Omega)$ up to the equivalence of norms, that is, whenever $0 \leq \beta < \frac{1}{4\omega}$, see (2.18). The operator A in (3.5) is now understood as the realization of $(-\Delta)^\omega$ in X^β.

Note that, due to the relation $u = \mathcal{R}^{\perp}\theta$, the following equality holds

$$u \cdot \nabla\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta) \cdot \left(\frac{\partial\theta}{\partial x_1}, \frac{\partial\theta}{\partial x_2}\right) = \nabla \cdot (u\theta) \quad \text{for } \theta \in D\left((-\Delta)^{\frac{1}{2}}\right) \subset H^1(\Omega). \tag{3.10}$$

A procedure, similar to the one that led to Theorem 3.4, allows us to find a local in time solution to (3.5) varying in the phase space $X^\alpha \hookrightarrow D\left((-\Delta)^{\omega\beta+\frac{1}{2}}\right)$ with α such that $\alpha \in \left[\beta + \frac{1}{2\omega}, \beta + 1\right)$ and $\alpha > \frac{1}{2\omega}$. Note that if $\beta > 0$ then we can take $\alpha = \beta + \frac{1}{2\omega}$. The key point is the Lipschitz continuity of the nonlinearity F on bounded sets of X^α into X^β, where $\alpha - \beta \in \left[\frac{1}{2\omega}, 1\right)$. Indeed, if $\theta_1, \theta_2 \in X^\alpha$ and $u_i = (u_{i1}, u_{i2}) = \mathcal{R}^{\perp}\theta_i$ for $i = 1, 2$, then

$$\begin{aligned}
\|\nabla \cdot (u_1\theta_1) - \nabla \cdot (u_2\theta_2)\|_\beta &\le \|\nabla \cdot (u_1(\theta_1 - \theta_2))\|_\beta + \|\nabla \cdot ((u_1 - u_2)\theta_2)\|_\beta \\
&\le c \sum_{j=1}^{2} \left(\|u_{1j}(\theta_1 - \theta_2)\|_{H^{1+2\omega\beta}(\Omega)} + \|(u_{1j} - u_{2j})\theta_2\|_{H^{1+2\omega\beta}(\Omega)}\right) \\
&\le c \sum_{j=1}^{2} \left(\|u_{1j}\|_{H^{2\omega\alpha}(\Omega)}\|\theta_1 - \theta_2\|_{H^{2\omega\alpha}(\Omega)} + \|u_{1j} - u_{2j}\|_{H^{2\omega\alpha}(\Omega)}\|\theta_2\|_{H^{2\omega\alpha}(\Omega)}\right) \\
&\le c \left(\|\theta_1\|_{D((-\Delta)^{\omega\alpha})} + \|\theta_2\|_{D((-\Delta)^{\omega\alpha})}\right) \|\theta_1 - \theta_2\|_{D((-\Delta)^{\omega\alpha})},
\end{aligned}$$

where we used Proposition 3.1, the property that ∇ is a bounded linear operator from $H^{1+2\omega\beta}(\Omega)$ into $H^{2\omega\beta}(\Omega)$ and the fact that $H^{2\omega\alpha}(\Omega)$ is a Banach algebra, since $2\omega\alpha > 1$, see [45, (1.89)]. Consequently, given $f \in X^\beta$, we have obtained

$$\|F(\theta_1) - F(\theta_2)\|_\beta \le c\,(\|\theta_1\|_\alpha + \|\theta_2\|_\alpha)\,\|\theta_1 - \theta_2\|_\alpha, \quad \theta_1, \theta_2 \in X^\alpha.$$

Theorem 3.6 *Let $0 \le \beta < \frac{1}{4\omega}$, $f \in X^\beta$ and $\alpha \in \left[\beta + \frac{1}{2\omega}, \beta + 1\right)$ be such that $\alpha > \frac{1}{2\omega}$. Then there exists a unique local in time (X^α, X^β) solution θ of (3.5) such that*

$$\theta \in C([0,\tau); D((-\Delta)^{\omega\alpha})) \cap C((0,\tau); D((-\Delta)^{\omega(\beta+1)})), \quad \theta_t \in C((0,\tau); D((-\Delta)^{\gamma})),$$

where $\gamma < \omega(\beta + 1)$.

In the above proposition, note that $\omega(\beta + 1) > \frac{3}{4}$ for β close to $\frac{1}{4\omega}$. Consequently, the constructed above local solutions starting at $\theta_0 \in D((-\Delta)^{\omega\alpha})$ regularize into $H^{\frac{3}{2}}(\Omega)$ for $t > 0$ as long as they exist.

Our next concern will be a priori estimates in smooth spaces, which yield global solutions of (3.4) at least for small data. The following lemma provides us a tool needed in the application of Lemma 2.28 to the Dirichlet problem (3.4) in a smoother base space.

Lemma 3.7 *For any $\omega \in \left(\frac{1}{2}, 1\right]$, $0 < \beta < \frac{1}{4\omega}$ and $r > \beta + \frac{1}{\omega} - \frac{1}{2}$ the following estimate of the nonlinear term in (3.5) is available*:

$$\|\nabla \cdot (u\theta)\|_\beta = \|(-\Delta)^{\omega\beta}[\nabla \cdot (u\theta)]\|_{L^2(\Omega)} \le c\|\theta\|_r^{2\nu}\|\theta\|_{\beta+\frac{1}{2}}^{2-2\nu}, \quad \theta \in X^r,$$

with some $0 < \nu < \frac{1}{2}$ and a positive constant c.

Proof We apply operator $(-\Delta)^{\omega\beta}$ with $\beta \in \left(0, \frac{1}{4\omega}\right)$ to the nonlinear term in (3.5), keeping the notation $u = \mathcal{R}^\perp\theta$ and invoking (3.10). We explicitly have

$$\|\nabla\cdot(u\theta)\|_\beta = \left\|(-\Delta)^{\omega\beta}\left[\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2}((-\Delta)^{-\frac{1}{2}}\theta)\theta\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial x_1}((-\Delta)^{-\frac{1}{2}}\theta)\theta\right)\right]\right\|_{L^2(\Omega)},$$

and proceed further with an estimate of the first term, using $D((-\Delta)^{\omega\beta}) = H^{2\omega\beta}(\Omega)$ (compare the characterization (2.18)),

$$\begin{aligned}
&\left\|(-\Delta)^{\omega\beta}\left[\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2}\left((-\Delta)^{-\frac{1}{2}}\theta\right)\theta\right)\right]\right\|_{L^2(\Omega)} \le c\left\|\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2}\left((-\Delta)^{-\frac{1}{2}}\theta\right)\theta\right)\right\|_{H^{2\omega\beta}(\Omega)} \\
&\le c\left\|\frac{\partial}{\partial x_2}\left((-\Delta)^{-\frac{1}{2}}\theta\right)\theta\right\|_{H^{1+2\omega\beta}(\Omega)} \le c\left\|\frac{\partial}{\partial x_2}\left((-\Delta)^{-\frac{1}{2}}\theta\right)\right\|_{H^{1+2\omega\beta}(\Omega)}\|\theta\|_{H^{1+2\omega\beta}(\Omega)} \\
&\le c\|\theta\|_{\beta+\frac{1}{2\omega}}^2,
\end{aligned} \tag{3.11}$$

where $H^{1+2\omega\beta}(\Omega)$ is a Banach algebra (see [45, (1.89)]), and an estimate in $H^{1+2\omega\beta}(\Omega)$ of the Riesz operator was used in the last inequality. The second term is treated analogously.

Since $r > \beta + \frac{1}{\omega} - \frac{1}{2}$ and $2\omega\left(\beta + \frac{1}{2}\right) \le 1 + 2\omega\beta < 2\omega r$, we also have by the moment inequality

$$\|\theta\|_{\beta+\frac{1}{2\omega}} \le c\|\theta\|_r^\nu\|\theta\|_{\beta+\frac{1}{2}}^{1-\nu}$$

with some $0 < \nu < \frac{1}{2}$. Therefore, we extend (3.11) to the form

$$\|\nabla \cdot (u\theta)\|_\beta \le c\|\theta\|_r^{2\nu}\|\theta\|_{\beta+\frac{1}{2}}^{2-2\nu},$$

which proves the claim. □

Since for $\beta > 0$ the available a priori estimate in $L^\infty(\Omega)$ is not sufficiently strong to guarantee global solvability of the problem, we will establish below an a priori estimate in $H^{2\omega\left(\beta+\frac{1}{2}\right)}(\Omega)$ for small data via Lemma 2.28. This will be carried out only for the values of parameter ω in (3.4) from the interval $\left(\frac{2}{3}, 1\right]$. An appropriate subordination condition will guarantee that these solutions exist globally in time.

Proposition 3.8 *Let $\omega \in \left(\frac{2}{3}, 1\right]$, $0 < \beta < \frac{1}{4\omega}$ and $\alpha \in \left(\beta + \frac{1}{\omega} - \frac{1}{2}, \beta + 1\right)$. If θ_0 is sufficiently small in the norm of $X^{\beta+\frac{1}{2}}$ and f is sufficiently small in the norm of X^β, then the (X^α, X^β) solution of (3.5) is bounded in $X^{\beta+\frac{1}{2}}$. In consequence, this solution exists globally in time.*

Proof The (X^α, X^β) solution θ of (3.5) defined on the maximal interval of existence was constructed in Theorem 3.6. Note that $\alpha > \beta + \frac{1}{2}$. Since $\omega \in \left(\frac{2}{3}, 1\right]$, we can take $r = \beta + 1$ in Lemma 3.7 and get

$$\left\| \mathcal{R}^\perp \theta \cdot \nabla \theta \right\|_\beta \le c \|\theta\|_{\beta+1}^{1-\varepsilon} \|\theta\|_{\beta+\frac{1}{2}}^{1+\varepsilon}, \quad \theta \in X^{\beta+1},$$

with some $\varepsilon \in (0, 1)$. Thus by Lemma 2.28 the solution is bounded in the norm of $X^{\beta+\frac{1}{2}}$ provided that the data is small. Applying Lemma 3.7 with $r = \alpha$, we obtain the following subordination condition

$$\|F(\theta)\|_\beta \le \left(c \, \|\theta\|_{\beta+\frac{1}{2}}^{1+\varepsilon} + \|f\|_\beta \right) \left(1 + \|\theta\|_\alpha^{1-\varepsilon} \right),$$

which by Theorem 2.29 implies that the solution θ exists globally in time. □

3.3 Solutions of (3.4) in a Base Space Larger than $L^2(\Omega)$

We begin with a simple observation helpful in deriving a range of parameters in the estimates made in this subsection and later in Sect. 4.3.

Lemma 3.9 *Let $a < b$, $c < d$ and $0 < b \le k$, $0 < d \le k$. Then there exist $x \in [a, b]$ and $y \in [c, d]$ such that $x + y = k$ and $x, y > 0$ if and only if $a + c \le k \le b + d$.*

The 2D quasi-geostrophic equation (3.4) with $\omega \in \left(\frac{1}{2}, 1\right]$ is now considered in one of the spaces X^β for $\beta \in [-1, 0)$ from the extrapolated fractional power scale generated by the self-adjoint operator $A = (-\Delta)^\omega$ in $L^2(\Omega)$. According to their characterization, the space X^β for $\beta \in [-1, 0)$ is isometric to $[D((-\Delta)^{-\omega\beta})]^*$. We also have

$$\mathcal{R}^\perp \varphi \cdot \nabla \psi = \nabla \cdot (\mathcal{R}^\perp \varphi \psi), \quad \varphi, \ \psi \in C_0^\infty(\Omega).$$

Note that for $\beta \in \left(-1, -\frac{1}{2\omega}\right]$

$$\frac{\partial}{\partial x_i} (-\Delta)^{\omega\beta} \colon L^2(\Omega) \to H^{-2\omega\beta-1}(\Omega) \hookrightarrow L^{\frac{1}{1+\omega\beta}}(\Omega)$$

is a bounded operator. Following the argument of [22, Lemma 2.2], for $\beta \in \left(-1, -\frac{1}{2\omega}\right]$ and $\phi \in C_0^\infty(\Omega)$, we get

$$\begin{aligned} \left\| \frac{\partial}{\partial x_i} \phi \right\|_\beta &= \left\| (-\Delta)^{\omega\beta} \frac{\partial}{\partial x_i} \phi \right\|_{L^2(\Omega)} = \max \left\{ \left| \left\langle v, (-\Delta)^{\omega\beta} \frac{\partial}{\partial x_i} \phi \right\rangle \right| : \ \|v\|_{L^2(\Omega)} \le 1 \right\} \\ &= \max \left\{ \left| \left\langle -\frac{\partial}{\partial x_i} (-\Delta)^{\omega\beta} v, \phi \right\rangle \right| : \ \|v\|_{L^2(\Omega)} \le 1 \right\} \le M \, \|\phi\|_{L^{-\frac{1}{\omega\beta}}(\Omega)}. \end{aligned}$$

Considering further $\phi = \mathcal{R}_j\varphi\psi$ with $\varphi, \psi \in C_0^\infty(\Omega)$ and $\alpha, \gamma > 0$, we estimate

$$\begin{aligned}\left\|\mathcal{R}_j\varphi\psi\right\|_{L^{-\frac{1}{\omega\beta}}(\Omega)} &\le \left\|\mathcal{R}_j\varphi\right\|_{L^p(\Omega)} \|\psi\|_{L^q(\Omega)} \le c\left\|\mathcal{R}_j\varphi\right\|_{H^{2\omega\alpha}(\Omega)} \|\psi\|_{H^{2\omega\gamma}(\Omega)}\\ &\le c\,\|\varphi\|_\alpha\,\|\psi\|_\gamma\,,\end{aligned}$$

which, by Lemma 3.9, holds with some $p, q \ge 2$ such that $\frac{1}{p} + \frac{1}{q} = -\omega\beta$ if and only if $-\omega\beta \ge 1 - \omega(\alpha+\gamma)$, that is,

$$\alpha - \beta + \gamma \ge \frac{1}{\omega}.$$

Thus for a fixed $\varphi \in X^\alpha$, $\alpha > 0$, the operator $P\colon \psi \mapsto \mathcal{R}^\perp\varphi \cdot \nabla\psi$ is a bounded operator from X^γ into X^β with its norm estimated by a multiple of $\|\varphi\|_\alpha$ provided that $\gamma > 0$ and $\alpha + \gamma - \beta \ge \frac{1}{\omega}$, i.e.,

$$\left\|\mathcal{R}^\perp\varphi\cdot\nabla\psi\right\|_\beta \le c\,\|\varphi\|_\alpha\,\|\psi\|_\gamma\,,\quad \varphi\in X^\alpha,\quad \psi\in X^\gamma \quad \text{if } \alpha-\beta+\gamma\ge\frac{1}{\omega},\quad \alpha,\gamma>0. \tag{3.12}$$

In particular, $P \in \mathcal{L}\left(X^\delta, X^{-\frac{1}{2\omega}}\right)$ for $\alpha + \delta \ge \frac{1}{2\omega}$, $\alpha, \delta > 0$, with its norm estimated by a multiple of $\|\varphi\|_\alpha$.

Observe now that for $\alpha, \delta > 0$ and $\varphi, \psi \in C_0^\infty(\Omega)$ (a dense subspace of X^α and $X^{\delta+\frac{1}{2\omega}}$), the estimate

$$\left\|\mathcal{R}_j\varphi\frac{\partial}{\partial x_i}\psi\right\|_{L^2(\Omega)} \le c\left\|\mathcal{R}_j\varphi\right\|_{L^p(\Omega)}\|\psi\|_{W^{1,q}(\Omega)} \le c\,\|\varphi\|_\alpha\,\|\psi\|_{\delta+\frac{1}{2\omega}}$$

holds with some $p, q \ge 2$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ provided that $\alpha + \delta \ge \frac{1}{2\omega}$, again by Lemma 3.9. In other words, for a fixed $\varphi \in X^\alpha$, $\alpha > 0$, the operator $P\colon \psi \mapsto \mathcal{R}^\perp\varphi\cdot\nabla\psi$ is bounded from $X^{\delta+\frac{1}{2\omega}}$ into $X^0 = L^2(\Omega)$ with its norm estimated by a multiple of $\|\varphi\|_\alpha$ provided that $\delta > 0$ and $\alpha + \delta \ge \frac{1}{2\omega}$.

Let $\beta \in \left(-\frac{1}{2\omega}, 0\right)$ and $\alpha + \delta \ge \frac{1}{2\omega}$, $\alpha, \delta > 0$. By the interpolation theory (see [32, Theorem 2.6]) and using the two earlier estimates, P is a bounded operator from $\left[X^\delta, X^{\delta+\frac{1}{2\omega}}\right]_{2\omega\beta+1}$ into $\left[X^{-\frac{1}{2\omega}}, X^0\right]_{2\omega\beta+1}$ with its norm estimated by a multiple of $\|\varphi\|_\alpha$. Using [32, p. 55] and [43, Theorem 1.18.10], we compute

$$\begin{aligned}[X^{-\frac{1}{2\omega}}, X^0]_{2\omega\beta+1} &= \left[\left[D\left((-\Delta)^{\frac{1}{2}}\right)\right]^*, L^2(\Omega)\right]_{2\omega\beta+1}\\ &= \left(\left[L^2(\Omega), D\left((-\Delta)^{\frac{1}{2}}\right)\right]_{-2\omega\beta}\right)^* = [D((-\Delta)^{-\omega\beta})]^* = X^\beta,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\left[X^\delta, X^{\delta+\frac{1}{2\omega}}\right]_{2\omega\beta+1} &= \left[D((-\Delta)^{\omega\delta}), D\left((-\Delta)^{\frac{1}{2}+\omega\delta}\right)\right]_{2\omega\beta+1} = D\left((-\Delta)^{\frac{1}{2}+\omega(\beta+\delta)}\right)\\ &= X^{\beta+\delta+\frac{1}{2\omega}}.\end{aligned} \tag{3.14}$$

Considering $\gamma > 0$ such that $\gamma - \beta > \frac{1}{2\omega}$ and taking $\delta = \gamma - \beta - \frac{1}{2\omega}$ in the above calculations, we also obtain (3.12) for $\beta \in \left(-\frac{1}{2\omega}, 0\right]$.

Corollary 3.10 *Let $\beta \in (-1, 0]$, $\alpha > 0$ and $\gamma > 0$. Then*

$$\left\| \mathcal{R}^{\perp}\varphi \cdot \nabla\psi \right\|_{\beta} \leq c \left\|\varphi\right\|_{\alpha} \left\|\psi\right\|_{\gamma}, \quad \varphi \in X^{\alpha}, \quad \psi \in X^{\gamma}$$

if $\alpha - \beta + \gamma \geq \frac{1}{\omega}$ and $\gamma - \beta > \frac{1}{2\omega}$.

As a consequence of Corollary 3.10, if $f \in X^{\beta}$ we can consider $F(\theta) = -\kappa^{-1}\mathcal{R}^{\perp}\theta \cdot \nabla\theta + f$ as a map from X^{α} into X^{β} with $\alpha > 0$, $\beta \in (-1, 0]$ satisfying $2\alpha - \beta \geq \frac{1}{\omega}$ and $\alpha - \beta > \frac{1}{2\omega}$. In this case, we obtain

$$\left\|F(\theta_1) - F(\theta_2)\right\|_{\beta} \leq c(\left\|\theta_1\right\|_{\alpha} + \left\|\theta_2\right\|_{\alpha}) \left\|\theta_1 - \theta_2\right\|_{\alpha}, \quad \theta_1, \theta_2 \in X^{\alpha}$$

which implies the local in time solvability of (3.5) in X^{β}.

Theorem 3.11 *Let $\omega \in \left(\frac{1}{2}, 1\right]$, $\beta \in \left(\frac{1}{\omega} - 2, 0\right]$ and $\alpha \in \left[\frac{1}{2}\beta + \frac{1}{2\omega}, \beta + 1\right)$ with $\alpha > \frac{1}{2\omega}$ if $\beta = 0$. Given $f \in X^{\beta}$, for each $\theta_0 \in X^{\alpha}$ there exists a unique (X^{α}, X^{β}) solution of (3.5),*

$$\theta \in C([0, \tau); X^{\alpha}) \cap C((0, \tau); X^{\beta+1}), \quad \theta_t \in C((0, \tau); X^{(\beta+1)-}),$$

defined on the maximal interval of existence and satisfying the first equation in (3.5) in X^{β} and the initial condition in (3.5) in X^{α}.

To examine the global existence in time of these solutions, we observe that a very similar reasoning as above leads to a suitable estimate of the bi-linear form associated with the nonlinearity, which involves the $L^{\infty}(\Omega)$ norm.

Proposition 3.12 *For $\beta \in (-1, 0]$ and $\gamma > 0$ such that $\gamma - \beta > \frac{1}{2\omega}$, we have*

$$\left\| \mathcal{R}^{\perp}\varphi \cdot \nabla\psi \right\|_{\beta} \leq c \left\|\varphi\right\|_{L^{\infty}(\Omega)} \left\|\psi\right\|_{\gamma}, \quad \varphi \in L^{\infty}(\Omega), \quad \psi \in X^{\gamma}.$$

We get the following subordination condition as a corollary.

Corollary 3.13 *For $\beta \in (-1, 0]$ and $\alpha > 0$ such that $\alpha - \beta > \frac{1}{2\omega}$, there exists $0 < \varepsilon < 1$ such that*

$$\left\| \mathcal{R}^{\perp}\theta \cdot \nabla\theta \right\|_{\beta} \leq c \left\|\theta\right\|_{L^{\infty}(\Omega)}^{1+\varepsilon} \left\|\theta\right\|_{\alpha}^{1-\varepsilon}, \quad \theta \in L^{\infty}(\Omega) \cap X^{\alpha}. \tag{3.15}$$

Proof Let $0 < \varepsilon < 1$ be such that $\alpha - \alpha\varepsilon - \beta > \frac{1}{2\omega}$. In view of Proposition 3.12, the left-hand side of (3.15) is estimated by a multiple of $\left\|\theta\right\|_{L^{\infty}(\Omega)} \left\|\theta\right\|_{\alpha(1-\varepsilon)}$. By the moment inequality (2.12), we also have

$$\left\|\theta\right\|_{\alpha(1-\varepsilon)} \leq c \left\|\theta\right\|_{L^2(\Omega)}^{\varepsilon} \left\|\theta\right\|_{\alpha}^{1-\varepsilon},$$

which yields (3.15). □

This allows us to extend the solutions globally in time.

Proposition 3.14 *If* $\omega \in \left(\frac{1}{2}, 1\right]$, $\beta \in \left(\frac{1}{2\omega} - 1, 0\right]$, $\frac{1}{2\omega} < \alpha < \beta + 1$ *and* $f \in L^\infty(\Omega)$, *then the* (X^α, X^β) *solutions of* (3.5) *constructed in Theorem* 3.11 *exist globally in time, i.e.,* $\tau = \infty$.

Proof The result is a consequence of the $L^\infty(\Omega)$ a priori bound (3.9) and Theorem 2.29 applied to (X^α, X^β) solutions of (3.5) from Theorem 3.11 with $\beta \in (\frac{1}{2\omega} - 1, 0]$ and $\frac{1}{2\omega} < \alpha < \beta + 1$. Indeed, for α, β from this range of parameters, we have $X^\alpha \hookrightarrow L^\infty(\Omega) = Y$ and (2.27) holds by Corollary 3.13. □

4 Fractional Chemotaxis System

Our second example is a *nonlocal model of chemotaxis*, which has recently been popularized by [3] and [44]. The considered problem is a generalization of the known Keller–Segel model of chemotaxis introduced in 1970 to describe the self-organization of a number of biological systems. In [4, 21] a motivation and mathematical background of the fractional version of that model was described and in [45, p. 243] a semigroup approach, similar to ours, to the original chemotaxis system was used. Due to the biological interpretation its solution $u(t,x)$ expressed density of the biological individuals (most often bacteria) at time t and location x and $v(t,x)$ denoted density of the chemical attractant produced to incline the colony to aggregate. Required by that interpretation, *non-negativity* of solutions was discussed e.g. in [45, p. 246] or [3, Sect. 2.4]. Concentrating here on mathematical aspects of the local and global solvability of (4.1), we consider solutions eventually changing their sign; non-negativity is thus an additional property required for solutions appearing in the biological application of that model. We will not study that property here.

Although typically considered as a Cauchy problem in $\mathbb{R}^N$, we study it here as a Dirichlet problem in a bounded C^2 domain Ω in the form

$$
\begin{aligned}
&u_t + (-\Delta)^\omega u + \nabla\cdot(u\nabla v) = 0, \quad x \in \Omega \subset \mathbb{R}^N, \quad t > 0,\\
&\Delta v + u = 0, \quad x \in \Omega,\ t > 0,\\
&u = v = 0 \ \text{ on } \ \partial\Omega,\\
&u(0,x) = u_0(x), \quad x \in \Omega,
\end{aligned} \tag{4.1}
$$

with a parameter $\omega \in \left(\frac{1}{2}, 1\right]$. Though at first sight this looks like a system of a parabolic and an elliptic equation, we will rewrite it as a single semilinear parabolic equation. Indeed, noting that

$$\nabla\cdot(u\nabla v) = \nabla u\cdot\nabla v + u\Delta v$$

and deriving (formally) v from the second equation of (4.1) as $v = (-\Delta)^{-1}u$, we insert it into the first equation of (4.1) to obtain the initial boundary value problem

$$\begin{aligned} &u_t + (-\Delta)^{\omega} u = -\nabla u \cdot \nabla((-\Delta)^{-1}u) + u^2, \\ &u = 0 \ \text{ on } \ \partial\Omega, \quad u(0,x) = u_0(x), \quad x \in \Omega. \end{aligned} \tag{4.2}$$

In this section we consider (4.2) as an abstract Cauchy problem

$$u_t + Au = F(u), \quad u(0) = u_0 \tag{4.3}$$

with $A = (-\Delta)^{\omega}$, $\omega \in \left(\frac{1}{2}, 1\right]$, being a positive sectorial operator in $L^p(\Omega)$ with a chosen p and

$$F(u) = -\nabla u \cdot \nabla((-\Delta)^{-1}u) + u^2,$$

where $-\Delta$ denotes the Dirichlet Laplacian realized in $L^p(\Omega)$. For the description of the domains of its fractional powers $(-\Delta)^{\sigma}$, $\sigma > 0$, see (2.19). Note also that, by (2.19), if u is in $D((-\Delta)^{\sigma})$, $\sigma > 0$, then $v = (-\Delta)^{-1}u$ satisfies zero Dirichlet boundary condition, which justifies the reformulation of the problem (4.1).

4.1 Solutions of (4.3) in $L^p(\Omega)$ as the Base Space

We choose $p > N$ and consider (4.3) in the base space $X = L^p(\Omega)$. We estimate the terms appearing in the nonlinearity F for any $\varphi, \psi \in W_0^{1,p}(\Omega)$ as follows

$$\begin{aligned} \|\nabla\varphi \cdot \nabla((-\Delta)^{-1}\psi)\|_{L^p(\Omega)} &\le c\|\varphi\|_{W^{1,p}(\Omega)}\|(-\Delta)^{-1}\psi\|_{W^{1,\infty}(\Omega)} \\ &\le c\|\varphi\|_{W^{1,p}(\Omega)}\|(-\Delta)^{-1}\psi\|_{W^{2,p}(\Omega)} \le c\|\varphi\|_{W_0^{1,p}(\Omega)}\|\psi\|_{L^p(\Omega)}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} \|\varphi\psi\|_{L^p(\Omega)} &\le \|\varphi\|_{L^{2p}(\Omega)}\|\psi\|_{L^{2p}(\Omega)} \\ &\le c\|\varphi\|_{L^{\infty}(\Omega)}\|\psi\|_{L^{\infty}(\Omega)} \le c\|\varphi\|_{W_0^{1,p}(\Omega)}\|\psi\|_{W_0^{1,p}(\Omega)}. \end{aligned} \tag{4.5}$$

These estimates show that both above terms are bi-linear operators from $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ into $L^p(\Omega)$. Hence the nonlinearity F is Lipschitz continuous on bounded sets as an operator between $X^{\alpha} = D((-\Delta)^{\omega\alpha}) \hookrightarrow W_0^{1,p}(\Omega)$ with $\alpha \ge \frac{1}{2\omega}$ and $X = L^p(\Omega)$. Indeed, given $u_1, u_2 \in B$ with B bounded in X^{α}, $\alpha \ge \frac{1}{2\omega}$, by (4.4) and (4.5) we have

$$\begin{aligned} \|F(u_1) - F(u_2)\|_{L^p(\Omega)} \le\ &\|\nabla(u_1 - u_2) \cdot \nabla((-\Delta)^{-1}u_1)\|_{L^p(\Omega)} \\ &+ \|\nabla u_2 \cdot \nabla((-\Delta)^{-1}(u_1 - u_2))\|_{L^p(\Omega)} \\ &+ \|u_1 - u_2\|_{W_0^{1,p}(\Omega)}\|u_1 + u_2\|_{W_0^{1,p}(\Omega)} \\ \le\ & c\|u_1 - u_2\|_{W_0^{1,p}(\Omega)}\left(\|u_1\|_{W_0^{1,p}(\Omega)} + \|u_2\|_{W_0^{1,p}(\Omega)}\right) \le c_B\|u_1 - u_2\|_{\alpha}. \end{aligned}$$

Consequently, we get local solvability of (4.3) in the phase space X^α for $\frac{1}{2\omega} \le \alpha < 1$.

Theorem 4.1 *Assume that* $\omega \in \left(\frac{1}{2}, 1\right]$, $\partial\Omega \in C^2$ *and* $u_0 \in X^\alpha$ *with* $\frac{1}{2\omega} \le \alpha < 1$. *Then there exists a unique local* (X^α, X^0) *solution to the problem* (4.3) *having the following regularity properties*:

$$u \in C([0,\tau); D((-\Delta)^{\omega\alpha})) \cap C((0,\tau); D((-\Delta)^{\omega})), \quad u_t \in C((0,\tau); D((-\Delta)^{\gamma})),$$

where $\gamma < \omega$ *and* $\tau > 0$ *stands for the maximal time of existence of that solution. Moreover, the corresponding Duhamel formula is satisfied in* X.

Remark 4.2 Considering the problem in dimension $N = 3$, we can stay within the fractional scale of Hilbert spaces defined by $A = (-\Delta)^\omega$, where $-\Delta$ is the negative Laplacian realized in the base space $X = L^2(\Omega)$. Indeed, this observation is a simple consequence of $-\Delta$ being an isometry between $D\left((-\Delta)^{\frac{3}{2}}\right)$ and $D\left((-\Delta)^{\frac{1}{2}}\right)$, see (2.9), and the following estimates valid for $\varphi, \psi \in H_0^1(\Omega) = D\left((-\Delta)^{\frac{1}{2}}\right)$

$$\begin{aligned}
\|\nabla\varphi \cdot \nabla((-\Delta)^{-1}\psi)\|_{L^2(\Omega)} &\le c\|\varphi\|_{H_0^1(\Omega)}\|(-\Delta)^{-1}\psi\|_{W^{1,\infty}(\Omega)} \\
&\le c\|\varphi\|_{H_0^1(\Omega)}\|(-\Delta)^{-1}\psi\|_{H^3(\Omega)} \le c\|\varphi\|_{H_0^1(\Omega)}\|\psi\|_{H_0^1(\Omega)}, \\
\|\varphi\psi\|_{L^2(\Omega)} \le \|\varphi\|_{L^4(\Omega)}\|\psi\|_{L^4(\Omega)} &\le c\|\varphi\|_{H_0^1(\Omega)}\|\psi\|_{H_0^1(\Omega)}.
\end{aligned}$$

Hence the nonlinear terms give rise to bi-linear transformations from $H_0^1(\Omega) \times H_0^1(\Omega)$ into $L^2(\Omega)$ and F satisfies in this case for $u_1, u_2 \in H_0^1(\Omega)$

$$\|F(u_1) - F(u_2)\|_{L^2(\Omega)} \le c(\|u_1\|_{H_0^1(\Omega)} + \|u_2\|_{H_0^1(\Omega)})\, \|u_1 - u_2\|_{H_0^1(\Omega)}\,. \tag{4.6}$$

Consequently, if $N = 3$ we again obtain local solvability of the problem (4.3) in the phase space $X^\alpha = D((-\Delta)^{\omega\alpha})$ with $\frac{1}{2\omega} \le \alpha < 1$, which is a Hilbert space embedded into $H_0^1(\Omega)$, and Theorem 4.1 holds true if $X = L^2(\Omega)$ and $N = 3$.

In case of the Cauchy problem in $\mathbb{R}^N$ for chemotaxis equation a *singular steady state* or a *generalized Chandrasekhar type solution* $u_C(x)$ exists, see [3, p. 32]. It is further shown there that *to a nonnegative, radial and sufficiently regular initial data staying below* $u_C(x)$ *correspond global in time solutions*, cp. [3, Theorem 2.3.1]. Even stronger property is known to hold in 1D case for solutions of the fractional Keller–Segel system on the real line with $\omega \in \left(\frac{1}{2}, 1\right)$; all solutions are global in time in that case (see [21] for a more complete analysis).

Here, in case of the Dirichlet boundary condition, we will show that local (X^α, X^0) solutions with $\alpha \in \left[\frac{1}{2\omega}, 1\right)$, introduced in Remark 4.2 for $N = 3$, will be extended globally in time if their initial data come from a certain neighbourhood of zero in the space $X^{\frac{1}{2}} = D\left((-\Delta)^{\frac{\omega}{2}}\right)$. To this end, observe that for $u \in D((-\Delta)^{\omega}) = X^1$ we have

$$\begin{aligned}\left\|\nabla u \cdot \nabla((-\Delta)^{-1}u)\right\|_{L^2(\Omega)} &\le \|u\|_{H^1(\Omega)} \left\|(-\Delta)^{-1}u\right\|_{W^{1,\infty}(\Omega)}\\ &\le c\,\|u\|_{H^1(\Omega)} \left\|(-\Delta)^{-1}u\right\|_{H^{\omega+2}(\Omega)} \le c\,\|u\|_{H^1(\Omega)}\,\|u\|_{D\left((-\Delta)^{\frac{\omega}{2}}\right)}\\ &\le c\,\|u\|^{\frac{1}{2\omega}}_{H^{2\omega}(\Omega)}\,\|u\|^{2-\frac{1}{2\omega}}_{D\left((-\Delta)^{\frac{\omega}{2}}\right)},\\ \left\|u^2\right\|_{L^2(\Omega)} = \|u\|^2_{L^4(\Omega)} \le c\,\|u\|^2_{H^{\frac{3}{4}}(\Omega)} &\le c\,\|u\|^{\frac{1}{2\omega}}_{H^{2\omega}(\Omega)}\,\|u\|^{2-\frac{1}{2\omega}}_{D\left((-\Delta)^{\frac{\omega}{2}}\right)},\end{aligned}$$

where the following consequences of the Nirenberg–Gagliardo inequality were used

$$\|u\|_{H^1(\Omega)} \le c\,\|u\|^{\frac{1}{2\omega}}_{H^{2\omega}(\Omega)}\,\|u\|^{1-\frac{1}{2\omega}}_{H^{\omega}(\Omega)}, \quad \|u\|_{H^{\frac{3}{4}}(\Omega)} \le c\,\|u\|^{\frac{1}{4\omega}}_{H^{2\omega}(\Omega)}\,\|u\|^{1-\frac{1}{4\omega}}_{H^{\omega}(\Omega)}.$$

Thus we obtain

$$\|F(u)\|_{L^2(\Omega)} \le c\,\|u\|_1^{\frac{1}{2\omega}}\,\|u\|_{\frac{1}{2}}^{2-\frac{1}{2\omega}}, \quad u \in X^1,$$

which means that the assumption (2.21) is satisfied with $\beta = 0$ and $b = 0$. Therefore, it follows from Lemma 2.28 that for $\alpha \in \left[\frac{1}{2\omega}, 1\right)$ the (X^α, X^0) solutions of (4.3) originating in a sufficiently small ball around zero in $X^{\frac{1}{2}} = D\left((-\Delta)^{\frac{\omega}{2}}\right)$ are bounded a priori in $X^{\frac{1}{2}}$.

Remark 4.3 If $\omega = 1$ the a priori estimate in $X^{\frac{1}{2}} = H^1_0(\Omega)$ is an immediate consequence of the quadratic estimate

$$\|F(u)\|_{L^2(\Omega)} \le c\|u\|^2_{\frac{1}{2}}, \tag{4.7}$$

see (4.6), and a simplified argument of the proof of Lemma 2.28. Then the (X^α, X^0) solutions with $\alpha \in \left[\frac{1}{2}, 1\right)$, which correspond to small initial data satisfying

$$\|u_0\|^2_{H^1_0(\Omega)} \le \frac{\lambda_1}{c^2},$$

where c is the constant at the right hand side of (4.7) and λ_1 denotes the first eigenvalue of $-\Delta$, are bounded a priori in $H^1_0(\Omega)$. Consequently, they exist globally in time.

In order to prove global extendibility in time of (X^α, X^0) solutions of (4.3) for $\alpha \in \left(\frac{1}{2\omega}, 1\right)$ and small initial data, it suffices to show a subordination condition involving the $X^{\frac{1}{2}}$ norm in which these solutions are bounded a priori.

Note that if $\alpha \in \left(\frac{1}{2\omega}, 1\right)$ then for $u \in X^\alpha$

$$\left\|\nabla u \cdot \nabla((-\Delta)^{-1}u)\right\|_{L^2(\Omega)} \le c\,\|u\|_{H^1(\Omega)}\,\|u\|_{D\left((-\Delta)^{\frac{\omega}{2}}\right)} \le c\,\|u\|^{1-\delta}_{\alpha}\,\|u\|^{1+\delta}_{\frac{1}{2}},$$

with any $0 < \delta < \frac{2\omega\alpha-1}{\omega(2\alpha-1)} \le 1$, since by the Nirenberg–Gagliardo inequality we have

$$\|u\|_{H^1(\Omega)} \le c\, \|u\|^{1-\delta}_{H^{2\omega\alpha}(\Omega)}\, \|u\|^{\delta}_{H^{\omega}(\Omega)}\,.$$

Moreover, we also have

$$\left\|u^2\right\|_{L^2(\Omega)} \le c\, \|u\|^2_{H^{\frac{3}{4}}(\Omega)} \le c\, \|u\|^{1-\delta}_{\alpha}\, \|u\|^{1+\delta}_{\frac{1}{2}}\,,$$

since by the Nirenberg–Gagliardo inequality we get

$$\|u\|_{H^{\frac{3}{4}}(\Omega)} \le c\, \|u\|^{\frac{1}{2}-\frac{\delta}{2}}_{H^{2\omega\alpha}(\Omega)}\, \|u\|^{\frac{1}{2}+\frac{\delta}{2}}_{H^{\omega}(\Omega)}$$

with δ as above. Concluding, we obtain

$$\|F(u)\|_{L^2(\Omega)} \le c\, \|u\|^{1+\delta}_{\frac{1}{2}}\, \|u\|^{1-\delta}_{\alpha}\,, \quad u \in X^{\alpha},$$

which is a form of a subordination condition as in (2.27). Hence by Theorem 2.29 the local (X^{α}, X^0) solutions of the fractional chemotaxis equation (4.3) exist globally in time for initial data with small $X^{\frac{1}{2}}$ norm.

Note that for $\alpha = \frac{1}{2\omega}$ we have the estimate

$$\|F(u)\|_{L^2(\Omega)} \le c\, \|u\|_{\alpha}\, \|u\|_{\frac{1}{2}}\,, \quad u \in X^{\alpha},$$

which combined with the boundedness of the $X^{\frac{1}{2}}$ norm of solutions with small initial data implies that F is sublinear in this case. The Duhamel formula and the Volterra type inequality guarantee that these local solutions can be extended globally in time.

Corollary 4.4 *Let $\omega \in \left(\frac{1}{2}, 1\right]$, $N = 3$ and $X = L^2(\Omega)$. Then the (X^{α}, X^0) solutions of (4.3) with $\alpha \in \left[\frac{1}{2\omega}, 1\right)$ from Remark 4.2 exist globally in time if their initial conditions $u_0 \in X^{\alpha}$ are taken from a sufficiently small neighborhood of zero in $X^{\frac{1}{2}}$.*

4.2 More Regular Local Solutions of (4.3)

If $-\Delta$ is realized in the space $L^p(\Omega)$ with $p > N$, we are also able to construct more regular local solutions to the fractional chemotaxis problem (4.3) by considering it in the base space $X^{\beta} = D((-\Delta)^{\omega\beta}) = W^{2\omega\beta,p}(\Omega)$ with $\beta \in \left[0, \frac{1}{2\omega p}\right)$, see (2.19).

For this purpose, we will estimate the bi-linear form $\nabla\varphi \cdot \nabla((-\Delta)^{-1}\psi)$ in the $D((-\Delta)^{\sigma})$ norm for $0 \le \sigma < \frac{1}{2p}$, that is, in the case when the fractional power space does not involve the boundary condition. To shorten some notations, we set $g := \nabla(-\Delta)^{-1}\psi$ and $h := \nabla\varphi$. We need to estimate $\|g \cdot h\|_{D((-\Delta)^{\sigma})}$ when $0 \le \sigma < \frac{1}{2p}$ for $p > N$. We have

$$\|g \cdot h\|_{D((-\Delta)^{\sigma})} = \|g \cdot h\|_{W^{2\sigma,p}(\Omega)} \le c\|g \cdot h\|_{W^{1,p}(\Omega)}$$

and estimate the $W^{1,p}(\Omega)$ norm, that is,

$$\|g \cdot h\|_{W^{1,p}(\Omega)} \leq \|g \cdot h\|_{L^p(\Omega)} + \sum_i \left\| \frac{\partial}{\partial x_i}(g \cdot h) \right\|_{L^p(\Omega)}.$$

We have

$$\begin{aligned} \|g \cdot h\|_{L^p(\Omega)} &= \|\nabla((-\Delta)^{-1}\psi) \cdot \nabla\varphi\|_{L^p(\Omega)} \leq c\|(-\Delta)^{-1}\psi\|_{W^{1,\infty}(\Omega)}\|\varphi\|_{W^{1,p}(\Omega)} \\ &\leq c\|(-\Delta)^{-1}\psi\|_{W^{2,p}(\Omega)}\|\varphi\|_{W^{1,p}(\Omega)} \leq c\|\varphi\|_{W^{1,p}(\Omega)}\|\psi\|_{L^p(\Omega)}, \end{aligned}$$

whereas the second term is estimated as follows:

$$\begin{aligned} \left\| \frac{\partial}{\partial x_i}(g \cdot h) \right\|_{L^p(\Omega)} &\leq \left\| \frac{\partial g}{\partial x_i} \cdot h \right\|_{L^p(\Omega)} + \left\| g \cdot \frac{\partial h}{\partial x_i} \right\|_{L^p(\Omega)} \\ &\leq \|g\|_{W^{1,p}(\Omega)}\|h\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Omega)}\|h\|_{W^{1,p}(\Omega)} \\ &\leq c\|(-\Delta)^{-1}\psi\|_{W^{2,p}(\Omega)}\|\varphi\|_{W^{1,\infty}(\Omega)} + c\|(-\Delta)^{-1}\psi\|_{W^{1,\infty}(\Omega)}\|\varphi\|_{W^{2,p}(\Omega)} \\ &\leq c\|\varphi\|_{W^{2,p}(\Omega)}\|\psi\|_{L^p(\Omega)}. \end{aligned}$$

We also get

$$\|\varphi\psi\|_{L^p(\Omega)} \leq \|\varphi\|_{L^\infty(\Omega)}\|\psi\|_{L^p(\Omega)} \leq c\|\varphi\|_{W^{1,p}(\Omega)}\|\psi\|_{L^p(\Omega)}$$

and

$$\begin{aligned} \left\| \frac{\partial}{\partial x_i}(\varphi\psi) \right\|_{L^p(\Omega)} &\leq \|\varphi\|_{W^{1,\infty}(\Omega)}\|\psi\|_{L^p(\Omega)} + \|\varphi\|_{L^\infty(\Omega)}\|\psi\|_{W^{1,p}(\Omega)} \\ &\leq c\|\varphi\|_{W^{2,p}(\Omega)}\|\psi\|_{W^{1,p}(\Omega)}. \end{aligned}$$

Consequently, the nonlinearity defines a bi-linear form from $W^{2,p}(\Omega) \times W^{2,p}(\Omega)$ into $D((-\Delta)^\sigma)$, $0 \leq \sigma < \frac{1}{2p}$. Therefore, we obtain the following local solvability result.

Theorem 4.5 *Let* $\omega \in \left(1 - \frac{1}{2p}, 1\right]$, $\beta \in \left(\frac{1}{\omega} - 1, \frac{1}{2\omega p}\right)$ *and* $u_0 \in X^\alpha$ *with* $\alpha \in \left[\frac{1}{\omega}, \beta + 1\right)$. *Then there exists a unique solution* u *of the problem* (4.3) *such that*

$$u \in C([0, \tau); D((-\Delta)^{\omega\alpha})) \cap C((0, \tau); D((-\Delta)^{\omega\beta+\omega})), \quad u_t \in C((0, \tau); D((-\Delta)^{\omega\beta+\gamma})),$$

where $\gamma < \omega$ *and* $\tau = \tau_{u_0}$ *is the life time of this local solution. Moreover, the corresponding Duhamel formula is satisfied in* X^β.

4.3 Local Solutions of (4.3) in a Larger Base Space than $L^2(\Omega)$

Let $N \geq 2$, $\omega \in \left(\frac{1}{2}, 1\right]$ and Ω be a bounded C^2 domain in $\mathbb{R}^N$. Recall that the spaces X^β for $\beta \in [-1, 0)$ from the extrapolated scale generated by the self-adjoint operator

$A = (-\Delta)^{\omega}$ in $L^2(\Omega)$ are isometric to $[D((-\Delta)^{-\omega\beta})]^*$. We will estimate in X^{β} the bi-linear form $-\nabla \cdot (\varphi\nabla((-\Delta)^{-1}\psi))$ defined by the nonlinear term in the chemotaxis equation. Initially, we will consider $\beta \in \left(-1, -\frac{1}{2\omega}\right]$. For such β the operator

$$\frac{\partial}{\partial x_i}(-\Delta)^{\omega\beta} \colon L^2(\Omega) \to H^{-2\omega\beta-1}(\Omega) \hookrightarrow L^{s'}(\Omega)$$

is a bounded operator for $s' \geq 2$ such that $\frac{N+2+4\omega\beta}{2N} \leq \frac{1}{s'}$.

For $\beta \in \left(-1, -\frac{1}{2\omega}\right]$ and $\phi \in C_0^{\infty}(\Omega)$ (a dense subset of X^{α} for $\alpha > 0$), we get

$$\begin{aligned}\left\|\frac{\partial}{\partial x_i}\phi\right\|_{\beta} &= \left\|(-\Delta)^{\omega\beta}\frac{\partial}{\partial x_i}\phi\right\|_{L^2(\Omega)} = \max\left\{\left|\left\langle v, (-\Delta)^{\omega\beta}\frac{\partial}{\partial x_i}\phi\right\rangle\right| : \|v\|_{L^2(\Omega)} \leq 1\right\} \\ &= \max\left\{\left|\left\langle -\frac{\partial}{\partial x_i}(-\Delta)^{\omega\beta}v, \phi\right\rangle\right| : \|v\|_{L^2(\Omega)} \leq 1\right\} \leq M\|\phi\|_{L^s(\Omega)},\end{aligned}$$

where $\frac{1}{2} \leq \frac{1}{s} \leq \frac{N-2-4\omega\beta}{2N}$.

In the role of ϕ we consider $\phi = \varphi\frac{\partial}{\partial x_i}((-\Delta)^{-1}\psi)$ and estimate using the Hölder inequality and the action of $(-\Delta)^{\omega}$ on the extrapolated scale, see (2.14). We obtain

$$\begin{aligned}\|\nabla \cdot (\varphi\nabla((-\Delta)^{-1}\psi))\|_{\beta} &\leq M\sum_i \left\|\varphi\frac{\partial}{\partial x_i}((-\Delta)^{-1}\psi)\right\|_{L^s(\Omega)} \\ &\leq c\|\varphi\|_{L^p(\Omega)}\left\|(-\Delta)^{-1}\psi\right\|_{W^{1,q}(\Omega)} \leq c\|\varphi\|_{L^p(\Omega)}\left\|(-\Delta)^{-1}\psi\right\|_{\frac{N+2}{4\omega}-\frac{N}{2\omega q}} \\ &\leq c\|\varphi\|_{L^p(\Omega)}\|\psi\|_{\frac{N-2}{4\omega}-\frac{N}{2\omega q}} \leq c\|\varphi\|_{\alpha}\|\psi\|_{\gamma}\end{aligned}$$

with some $p, q \geq 2$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$ and $\alpha, \gamma > 0$ such that

$$2\omega\alpha - \frac{N}{2} \geq -\frac{N}{p} \quad \text{and} \quad 2\omega\gamma - \frac{N-2}{2} \geq -\frac{N}{q}.$$

By Lemma 3.9 such p and q exist if and only if

$$1 - \frac{1}{N} - \frac{2\omega}{N}(\alpha + \gamma) \leq \frac{1}{s} \leq 1.$$

Therefore, when $N \geq 2$, and for $\alpha, \gamma > 0$ such that

$$\alpha - \beta + \gamma \geq \frac{N}{4\omega},$$

we can choose $\frac{1}{s} = \frac{N-2-4\omega\beta}{2N}$.

Case 1. We have shown that if $N \geq 2$, $\beta \in \left(-1, -\frac{1}{2\omega}\right]$ and $\varphi \in X^{\alpha}$ with $\alpha > 0$, then the operator $P \colon \psi \mapsto -\nabla \cdot (\varphi\nabla((-\Delta)^{-1}\psi))$ is bounded from X^{γ} into X^{β} with its norm estimated by a multiple of $\|\varphi\|_{\alpha}$ provided that $\gamma > 0$ and $\alpha - \beta + \gamma \geq \frac{N}{4\omega}$:

$$\left\|\nabla\cdot(\varphi\nabla((-\Delta)^{-1}\psi))\right\|_{\beta}\leq c\,\|\varphi\|_{\alpha}\,\|\psi\|_{\gamma}\,,\quad \varphi\in X^{\alpha},\quad \psi\in X^{\gamma}$$
$$\text{if }\ \alpha-\beta+\gamma\geq\frac{N}{4\omega},\quad \alpha,\gamma>0. \tag{4.8}$$

In particular, taking $\beta=-\frac{1}{2\omega}$ in (4.8), we see that $P\in\mathcal{L}\left(X^{\delta},X^{-\frac{1}{2\omega}}\right)$ for $\alpha+\delta\geq\frac{N-2}{4\omega}$, $\alpha,\delta>0$, with its norm estimated by a multiple of $\|\varphi\|_{\alpha}$.

Case 2. Estimating now the bi-linear form in X^{β} with $\beta=0$, observe that for $\alpha,\delta>0$ and $\varphi,\psi\in C_0^{\infty}(\Omega)$ (a dense subspace of X^{α} and $X^{\delta+\frac{1}{2\omega}}$), we have

$$\begin{aligned}\left\|\nabla\varphi\cdot\nabla((-\Delta)^{-1}\psi)\right\|_{L^2(\Omega)}&\leq\|\varphi\|_{W^{1,p}(\Omega)}\left\|(-\Delta)^{-1}\psi\right\|_{W^{1,q}(\Omega)}\\&\leq c\,\|\varphi\|_{\alpha}\left\|(-\Delta)^{-1}\psi\right\|_{H^{2\omega\delta+3}(\Omega)}\leq c\,\|\varphi\|_{\alpha}\,\|\psi\|_{\delta+\frac{1}{2\omega}}\end{aligned}$$

with some $p,q\geq 2$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ provided that

$$2\omega\alpha-\frac{N}{2}\geq 1-\frac{N}{p}\quad\text{and}\quad 2\omega\delta+3-\frac{N}{2}\geq 1-\frac{N}{q},$$

which, by Lemma 3.9, means that $\alpha+\delta\geq\frac{N-2}{4\omega}$. We also have

$$\|\varphi\psi\|_{L^2(\Omega)}\leq\|\varphi\|_{L^{\tilde{p}}(\Omega)}\,\|\psi\|_{L^{\tilde{q}}(\Omega)}\leq c\,\|\varphi\|_{\alpha}\,\|\psi\|_{\delta+\frac{1}{2\omega}}$$

with some $\tilde{p},\tilde{q}\geq 2$ such that $\frac{1}{\tilde{p}}+\frac{1}{\tilde{q}}=\frac{1}{2}$ provided that

$$2\omega\alpha-\frac{N}{2}\geq-\frac{N}{\tilde{p}}\quad\text{and}\quad 2\omega\delta+1-\frac{N}{2}\geq-\frac{N}{\tilde{q}},$$

which, again by Lemma 3.9, also means that $\alpha+\delta\geq\frac{N-2}{4\omega}$.

Concluding, for a fixed $\varphi\in X^{\alpha}$, $\alpha>0$, the operator $P\colon\psi\mapsto-\nabla\varphi\cdot\nabla((-\Delta)^{-1}\psi)+\varphi\psi$ is bounded from $X^{\delta+\frac{1}{2\omega}}$ into $X^{0}=L^2(\Omega)$ with its norm estimated by a multiple of $\|\varphi\|_{\alpha}$ provided that $\delta>0$ and $\alpha+\delta\geq\frac{N-2}{4\omega}$.

Case 3. Let now $\beta\in\left(-\frac{1}{2\omega},0\right)$ and $\alpha+\delta\geq\frac{N-2}{4\omega}$, $\alpha,\delta>0$. Applying the interpolation theory (cp. [32, Theorem 2.6]) in the same way as for the 2D quasi-geostrophic equation [see (3.13) and (3.14)], it follows that P is a bounded operator from $X^{\beta+\delta+\frac{1}{2\omega}}$ into X^{β} with its norm estimated by a multiple of $\|\varphi\|_{\alpha}$. Thus for $\gamma>0$ such that $\gamma-\beta>\frac{1}{2\omega}$ we also obtain condition (4.8) for $\beta\in\left(-\frac{1}{2\omega},0\right)$ by setting $\delta=\gamma-\beta-\frac{1}{2\omega}$.

Corollary 4.6 *Let $N\geq 2$, $\beta\in(-1,0]$, $\alpha>0$ and $\gamma>0$. Then*

$$\left\|\nabla\cdot(\varphi\nabla((-\Delta)^{-1}\psi))\right\|_{\beta}\leq c\,\|\varphi\|_{\alpha}\,\|\psi\|_{\gamma}\,,\quad \varphi\in X^{\alpha},\quad \psi\in X^{\gamma},$$

if $\alpha - \beta + \gamma \geq \frac{N}{4\omega}$ *and* $\gamma - \beta > \frac{1}{2\omega}$.

As a consequence of Corollary 4.6, we can consider $F(u) = -\nabla \cdot (u\nabla((-\Delta)^{-1}u))$ as a map from X^α into X^β with $\alpha > 0$, $\beta \in (-1, 0]$ satisfying $2\alpha - \beta \geq \frac{N}{4\omega}$ and $\alpha - \beta > \frac{1}{2\omega}$. In this case, we obtain

$$\|F(u_1) - F(u_2)\|_\beta \leq c(\|u_1\|_\alpha + \|u_2\|_\alpha)\, \|u_1 - u_2\|_\alpha\,, \quad u_1, u_2 \in X^\alpha.$$

This Lipschitz condition on bounded subsets of X^α immediately yields the following local existence result.

Theorem 4.7 *Let* $N \geq 2$, $\beta \in (-1, 0]$ *and* $\alpha > 0$ *be such that* $2\alpha - \beta \geq \frac{N}{4\omega}$ *and* $\frac{1}{2\omega} < \alpha - \beta < 1$. *For each* $u_0 \in X^\alpha$ *there exists a unique* (X^α, X^β) *solution of the problem* (4.3),

$$u \in C([0, \tau); X^\alpha) \cap C((0, \tau); X^{\beta+1}), \quad u_t \in C((0, \tau); X^{(\beta+1)-}),$$

defined on the maximal interval of existence and satisfying the first equation in (4.3) *in* X^β *and the initial condition in* (4.3) *in* X^α. *Moreover, the corresponding Duhamel formula is satisfied in* X^β.

In particular, taking $\beta = 0$ and $N = 3$ in Theorem 4.7, we obtain the local (X^α, X^0) solutions of (4.3) for $\alpha \in \left(\frac{1}{2\omega}, 1\right)$ as already observed in Remark 4.2.

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